Pollution, 4(4): 745-758, Autumn 2018
DOI: 10.22059/poll.2018.253350.410
Print ISSN: 2383-451X Online ISSN: 2383-4501
Web Page: https://jpoll.ut.ac.ir, Email: jpoll@ut.ac.ir

Analytical Solutions of One-dimensional Advection Equation with Dispersion Coefficient as Function of Space in a Semi-infinite Porous Media

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Received: 24.02.2018 Accepted: 27.05.2018

ABSTRACT: The aim of this study is to develop analytical solutions for one-dimensional advection-dispersion equation in a semi-infinite heterogeneous porous medium. The geological formation is initially not solute free. The nature of pollutants and porous medium are considered non-reactive. Dispersion coefficient is considered squarely proportional to the seepage velocity where as seepage velocity is considered linearly spatially dependent. Varying type input condition for multiple point sources of arbitrary time-dependent emission rate pattern is considered at origin. Concentration gradient is considered zero at infinity. A new space variable is introduced by a transformation to reduce the variable coefficients of the advection-dispersion equation into constant coefficients. Laplace Transform Technique is applied to obtain the analytical solutions of governing transport equation. Obtain results are shown graphically for various parameter and value on the dispersion coefficient and seepage velocity. The developed analytical solutions may help as a useful tool for evaluating the aquifer concentration at any position and time.

Keywords: Advection, Dispersion, Unit step function, Point Source, Heterogeneous medium.

INTRODUCTION

Advection dispersion equation (ADE) is broadly used as governing equation to predict the transport phenomena in aquifer and groundwater (Bear, 1972). In order to deal aquifer contamination, it is necessary to infer the mechanism of mass transport in porous media. A large number of literatures are present to investigate solute transport in porous media. Most of the researchers have focused the solute transport distribution with point source pollutant in aquifer either in heterogeneous or homogeneous medium. Ogata and Banks (1961) obtained analytical solution to one-dimensional longitudinal transport while Harleman and Rumer (1963) derived for transverse spreading in the one-dimensional porous domain. Rumer (1962) obtained analytical solution, assuming dispersion coefficient directly related to flow velocity. Wierenga (1977) observed that variations (fluctuation) in velocity don't affect longitudinal dispersion in one-dimensional solute transport. DeSmedt and Wierenga (1978) observed that seepage flow under steady-state conditions in any geological formation always are temporally depends. Sauty (1980), Pickens and Grisak (1981) evoked that dispersion in geological formation is scale dependent. Sudicky and

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Cherry (1979) demonstrate that the dispersion enhance with distance from the solute source. Flury et al. (1998) obtained the analytical solution of the one-dimensional advection-dispersion equation with depth-dependent adsorption coefficients. Huang et al. (1996) obtained an analytical solution to conservative solute transport in heterogeneous porous media assuming dispersivity increases linearly with distance up to some distance after that it achieve asymptotic value.

Volocchi (1989) studied solute transport where sorption reactions directly related to an arbitrary function in upward direction. Guerrero et al. (2009) provided an exact solution of the advection-dispersion equation with constant coefficients using generalized integral Laplace transform technique. Chen et al. (2003) used a Laplace-transformed power series technique to solve a two-dimensional advection-dispersion equation in cylindrical coordinates and compared the solution with a numerical solution. Chen et al. (2008) obtained an analytical solution with an asymptotic hyperbolic dispersion coefficient. Singh et al. (2013) derived an analytical solution of the two-dimensional solute transport in a homogeneous porous medium using the Hankel transforms technique. Pang and Hunt (2001) obtained analytical solutions for advection dispersion equation with scale-dependent dispersion. Sanskrityayn et al. (2016) obtained analytical solution of advection dispersion equation with spatially and temporally dependent dispersion using Green's function while Longitudinal solute transport from a pulse type source along temporally and spatially dependent flow was discussed by Yadav et at. (2012). Kumar and Yadav (2015) obtained analytical solution of one-dimensional solute transport for uniform and varying pulse type input point source through heterogeneous porous medium. Das et al. (2017) presents mathematical modeling of groundwater contamination with varying velocity field while Moghaddam et al. (2017) developed a numerical model for one dimensional solute transport in rivers.

Aral and Liao (1996) obtained analytical solutions of the two-dimensional advection-dispersion equation with time-dependent dispersion coefficient. Massabo et al. (2006) developed analytical solutions for two-dimensional advection-dispersion equation with anisotropic dispersion. In the subsurface, flow and transport processes are mainly depending on spatial heterogeneity and temporal variability which occurs due to seasonal and variations in water levels (Elfeki et al., 2011).

The above literature review shows that the majority of the analytical solutions were mainly related to hypothesis in one and two dimensional ground-water flow in aquifers with common assumptions like constant porosity, steady and unsteady pore-water velocity with or without retardation factor. Almost all analytical solutions to any physical problem of subsurface involve complex boundary conditions to find the corresponding analytical solutions. Due to heterogeneity plumes moves at different rates because it generates variability in the fluid velocity. Most of the analytical/numerical solutions derived by pervious workers considered a point source of constant nature or time dependent.

The main focus of this paper is to derive a new mathematical model to investigating contaminant transport in an aquifer considering especially variable flow field. Deviating from previous studies, a multiple point source is considered to assess the impact of concentration level in groundwater contamination problems. The input condition is introduced at the origin of the domain and second condition is considered at the end of the domain. A new space variable is introduced by a transformation. It helps to reduce the variable coefficients of advection dispersion equation into constant coefficients. Laplace transformation technique is used to get the analytical solution which is more viable and simpler. The developed solutions may help to measure the contaminant concentration in an aquifer at any position and time.

MATERIAL & METHODS

The problem formulated mathematically as a multiple point source of one-dimensional semi-infinite geological formation which is initially not solute free. One-dimensional advection-dispersion equation (ADE) is used to formulate the present model which is mathematically written as follows:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x} - uC \right) \quad (1)$$

In which $C[\mathrm{ML}^{-3}]$ is the solute concentration of the pollutant, transporting along the flow field through the medium at a position $x[L]$ and time $t[\mathrm{T}]$. $D[\mathrm{L}^2\mathrm{T}^{-1}]$ and $u[\mathrm{LT}^{-1}]$ are the dispersion coefficient and unsteady uniform pore seepage velocity respectively. The first term of the left hand side of the Eq.(1) is represents change in concentration with time in liquid phase. The first term on the right-hand side of the Eq.(1) describes the influence of the dispersion on the concentration distribution while the second term is the change of the concentration due to advective transport. The medium is supposed to have a uniform solute concentration C_i before an injection of pollutant in the domain. The input condition is considered of varying type. The concentration gradient is assumed zero at right boundary. This type phenomenon mathematically may be written as:

$$C(x,t) = C_i \quad ; \quad t = 0\ ,\ x \geq 0 \quad (2)$$

$$-D\frac{\partial C}{\partial x} + uC = uC_0\left(pt^2 + qt + r\right) \left[u\left(t - t_1\right) - u\left(t - t_2\right)\right];\ x = 0, t > 0 \quad (3)$$

$$\frac{\partial C(x,t)}{\partial x} = 0 \quad ; \quad t \geq 0,\ x \to \infty \quad (4)$$

where, C_0 is the initial concentration, p, q and r are the parameters of the quadratic pulse boundary conditions at $x = 0$, t_1 and t_2 are the outset and terminating times of the source activation, respectively, where $u(t - t_i)$ is the shifted Heaviside function, which is 0 for $t < t_i$ and 1 for $t \geq t_i$. The geometry of the input boundary condition is shown in Figure (a).

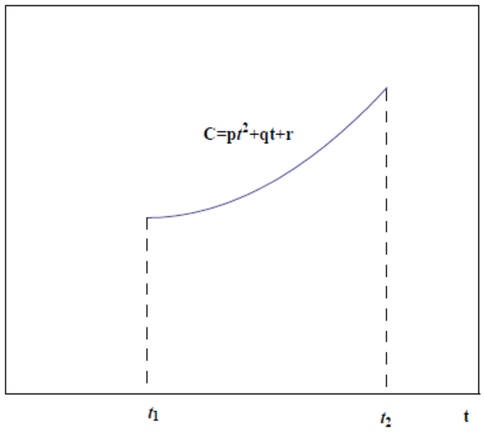

Fig. (a). Geometry of the input boundary condition

Freeze and Cherry (1979) proposed that dispersion is directly proportional to n[th] power of the seepage velocity where n[th] power varies from 1 to 2. In the present case, dispersion due to heterogeneity is considered directly proportional to the square of seepage velocity where as seepage velocity is considered as a linear function of space variable.

$$u = u_0(1 + a\,x) \text{ and } D \propto u^2 \Rightarrow D = D_0(1 + a\,x)^2 \quad (5)$$

where D_0 and u_0 are initial dispersion coefficient and seepage velocity respectively. $a\,[\mathrm{L}^{-1}]$ be the heterogeneity parameter whose dimension is the inverse of that of space variable (Kumar et al., 2010). The various values of a representing different heterogeneity. Heterogeneity of the porous medium means the transport properties like porosity or hydraulic conductivity is not uniform throughout the domain but depends upon the position.

Substituting values from Eq.(5) in Eq.(1), we have

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left\{D_0(1+ax)^2\frac{\partial C}{\partial x}-u_0(1+ax)C\right\} \quad (6)$$

Eqs.(2-4) may be written as:

$$C(x,t)=C_i \quad ; \quad t=0\,,\; x\geq 0 \quad (7)$$

$$-D_0\frac{\partial C}{\partial x}+u_0C=u_0C_0\left(pt^2+qt+r\right)\left[u\left(t-t_1\right)-u\left(t-t_2\right)\right];\; x=0, t>0 \quad (8)$$

$$\frac{\partial C(x,t)}{\partial x}=0 \quad ; \quad t\geq 0,\; x\to\infty \quad (9)$$

Let us introduce a new independent space variable X by a transformation (Kumar et al., (2010)) defined as:

$$X=\frac{log(1+ax)}{a} \quad \Rightarrow \quad \frac{dX}{dx}=\frac{1}{(1+ax)} \quad (10)$$

Applying the transformation of Eq. (10) on Eqs. (6-9), we have

$$\frac{\partial C}{\partial t}=D_0\frac{\partial^2 C}{\partial X^2}-U_0\frac{\partial C}{\partial X}-\gamma_0 C \quad (11)$$

where $U_0=(u_0-aD_0)$ and $\gamma_0=au_0$.

$$C(X,t)=C_i \quad ; \quad t=0\,,\; X\geq 0 \quad (12)$$

$$-D_0\frac{\partial C}{\partial X}+u_0C=u_0C_0\left(pt^2+qt+r\right)\left[u\left(t-t_1\right)-u\left(t-t_2\right)\right];\; X=0, t>0 \quad (13)$$

$$\frac{\partial C(X,t)}{\partial X}=0 \quad ; \quad t\geq 0,\; X\to\infty \quad (14)$$

Applying the Laplace transformation on above initial and boundary value problem, it reduces into an ordinary differential equation of second order, which comprises of following three equations:

$$D_0\frac{d^2\bar{C}}{dX^2}-U_0\frac{d\bar{C}}{dX}-(s+\gamma_0)\bar{C}=-C_i \quad (15)$$

where $\bar{C}=\int_0^\infty C(X,t)e^{-st}\,dt$

$$-D_0\frac{d\bar{C}}{dX}+u_0\bar{C}=u_0C_0\left[\left(pt_1^2+qt_1+r\right)\frac{\exp(-st_1)}{s}+(2pt_1+q)\frac{\exp(-st_1)}{s^2}+2p\frac{\exp(-st_1)}{s^3}-\left(pt_2^2+qt_2+r\right)\frac{\exp(-st_2)}{s}-(2pt_2+q)\frac{\exp(-st_2)}{s^2}-2p\frac{\exp(-st_2)}{s^3}\right];\quad X=0 \quad (16)$$

$$\frac{d\bar{C}}{dX}=0 \quad ; \quad X\to\infty \quad (17)$$

where s is a Laplace parameter.

Thus the general solution of ordinary differential equation (15) may be written as:

$$\bar{C}(X,s)=c_1 exp\left(\mu+X\sqrt{\alpha+\beta s}\right)+c_2 exp\left(\mu-X\sqrt{\alpha+\beta s}\right)+\frac{C_i}{(s+\gamma_0)} \quad (18)$$

where $\alpha=\frac{U_0^2}{4D_0^2}+\frac{\gamma_0}{D_0}$, $\beta=\frac{1}{D_0}$, $\mu=\frac{U_0X}{2D_0}$

Now, using boundary conditions Eq. (16) and (17) in general solution Eq. (18) to eliminate arbitrary constants c_1 and c_2, we get the particular solution to the above boundary value problem as:

$$\bar{C}(X,s)=\frac{C_i}{(s+\gamma_0)}-\frac{\beta u_0C_i\exp\left(\mu-X\sqrt{\alpha+\beta s}\right)}{(s+\gamma_0)\left(\sqrt{\alpha}+\sqrt{\alpha+\beta s}\right)}+\left\{\left(pt_1^2+qt_1+r\right)\frac{\exp(-st_1)}{s}+(2pt_1+q)\frac{\exp(-st_1)}{s^2}+2p\frac{\exp(-st_1)}{s^3}-\left(pt_2^2+qt_2+r\right)\frac{\exp(-st_2)}{s}-(2pt_2+q)\frac{\exp(-st_2)}{s^2}-2p\frac{\exp(-st_2)}{s^3}\right\}\frac{\beta u_0C_0\exp\left(\mu-X\sqrt{\alpha+\beta s}\right)}{(s+\gamma_0)\left(\sqrt{\alpha}+\sqrt{\alpha+\beta s}\right)} \quad (19)$$

Apply Inverse Laplace transformation on Eq.(19) and using the result given by Van Genuchten and Alves, (1982) and Abramowitz and Stegun, (1970). Using back transformations Eq.(10)., the analytical solutions of advection-dispersion equation for varying type input point source may be written in terms of $C(x,t)$ as:

$$C(x,t)=C_i\exp(-\gamma_0 t)-\beta u_0 C_i F(X,t) \quad ; \quad 0\le t<t_1 \tag{20}$$

$$C(x,t)=C_i\exp(-\gamma_0 t)-\beta u_0 C_i F(X,t)+\beta u_0 C_0\exp\left(\frac{U_0 X}{2D_0}\right)\Big\{\left(pt_1^2+qt_1+r\right) G(X,t-t_1)+(2pt_1+q)H(X,t-t_1)+2\,p\,J(X,t-t_1)\Big\} \quad ; \; t_1\le t<t_2 \tag{21}$$

$$C(x,t)=C_i\,exp(-\gamma_0 t)-\beta u_0 C_i F(X,t)+\beta u_0 C_0\,exp\left(\frac{U_0 X}{2D_0}\right)\Big\{\left(pt_1^2+qt_1+r\right) G(X,t-t_1)+(2pt_1+q)H(X,t-t_1)+2\,p\,J(X,t-t_1)-\left(pt_2^2+qt_2+r\right) G(X,t-t_2)-(2pt_2+q)H(X,t-t_2)-2\,p\,J(X,t-t_2)\Big\} \quad ; \quad t\ge t_2 \tag{22}$$

where

$$F(X,t)=\frac{\exp(-\delta X-\gamma_0 t)}{2\left(\delta+\sqrt{\alpha}\right)}erfc\left(\frac{\beta X-2\delta t}{2\sqrt{\beta t}}\right)-\frac{\exp(\delta X-\gamma_0 t)}{2\left(\delta-\sqrt{\alpha}\right)}erfc\left(\frac{\beta X+2\delta t}{2\sqrt{\beta t}}\right)+\frac{\sqrt{\alpha}\exp\left(X\sqrt{\alpha}\right)}{\left(\delta^2-\alpha\right)}erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$G(X,t)=\sqrt{\frac{t}{\pi\beta}}.\exp\left(-\frac{4\alpha t^2+\beta^2X^2}{4\beta t}\right)+\frac{1}{4\sqrt{\alpha}}\exp\left(-X\sqrt{\alpha}\right)erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right) -\frac{1}{4\sqrt{\alpha}}\left(1+2X\sqrt{\alpha}+\frac{4\alpha t}{\beta}\right)\exp\left(X\sqrt{\alpha}\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$H(X,t)=\frac{t}{4\alpha}\left(1+X\sqrt{\alpha}+\frac{2\alpha t}{\beta}\right)\sqrt{\frac{\beta}{\pi t}}\exp\left(-\frac{4\alpha t^2+\beta^2X^2}{4\beta t}\right)-\frac{\beta}{16\alpha\sqrt{\alpha}}\left(1+2X\sqrt{\alpha}-\frac{4\alpha t}{\beta}\right)\exp\left(-X\sqrt{\alpha}\right) erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)+\frac{\beta}{16\alpha\sqrt{\alpha}}\left\{1-\frac{2\alpha}{\beta^2}\left(\beta X+2t\sqrt{\alpha}\right)^2-\frac{4\alpha t}{\beta}\right\}\exp\left(X\sqrt{\alpha}\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$J(X,t)=\frac{1+X\sqrt{\alpha}}{4\alpha}\sqrt{\frac{\beta}{\pi}}J_1(X,t)+\frac{1}{2\sqrt{\beta\pi}}J_2(X,t)+\frac{\exp\left(-X\sqrt{\alpha}\right)}{4\alpha}J_3(X,t)-\frac{\beta\left(1+2X\sqrt{\alpha}\right)}{16\alpha\sqrt{\alpha}}J_4(X,t)\exp\left(-X\sqrt{\alpha}\right) -\frac{\exp\left(X\sqrt{\alpha}\right)}{4\alpha}J_5(X,t)+\frac{\beta\exp\left(X\sqrt{\alpha}\right)}{16\alpha\sqrt{\alpha}}J_6(X,t)-\frac{\exp\left(X\sqrt{\alpha}\right)}{8\beta\sqrt{\alpha}}J_7(X,t),$$

$$J_1(X,t)=-\frac{\beta\sqrt{t}}{\alpha}exp\left(-\frac{4\alpha t^2+\beta^2X^2}{4\beta t}\right)+\frac{\beta}{4\alpha}\sqrt{\frac{\pi\beta}{\alpha}}\Big\{\left(1+X\sqrt{\alpha}\right)exp\left(-X\sqrt{\alpha}\right) erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)-\left(1-X\sqrt{\alpha}\right)exp\left(X\sqrt{\alpha}\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)\Big\},$$

$$J_2(X,t)=-\frac{\beta\sqrt{t}}{2\alpha^2}(3\beta+2\alpha t)\exp\left(-\frac{4\alpha t^2+\beta^2X^2}{4\beta t}\right)+\frac{\beta^2}{8\alpha^2}\sqrt{\frac{\pi\beta}{\alpha}}\Big\{\left(3+3X\sqrt{\alpha}+X^2\alpha\right)\exp\left(-X\sqrt{\alpha}\right)erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right) -\left(3-3X\sqrt{\alpha}+X^2\alpha\right)\exp\left(X\sqrt{\alpha}\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)\Big\},$$

$$J_3(X,t)=\frac{\beta}{4\alpha}\sqrt{\frac{t\beta}{\pi\alpha}}\left(3+X\sqrt{\alpha}+\frac{2t\alpha}{\beta}\right)\exp\left\{-\frac{1}{t}\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta}}\right)^2\right\}+\frac{1}{16}\left(\frac{\beta}{\alpha}\right)^2\left(3-2X\sqrt{\alpha}\right)\exp\left(2X\sqrt{\alpha}\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)$$

$$-\frac{1}{16}\left(\frac{\beta}{\alpha}\right)^2\left(3+4X\sqrt{\alpha}+2X^2\alpha\right)erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)+\frac{t^2}{2}erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$J_4(X,t)=\sqrt{\frac{t\beta}{\pi\alpha}}\exp\left\{-\frac{1}{t}\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta}}\right)^2\right\}+\left(t-\frac{X\beta}{2\sqrt{\alpha}}-\frac{\beta}{4\alpha}\right)erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)+\frac{\beta}{4\alpha}\exp\left(2X\sqrt{\alpha}\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$J_5(X,t)=-\frac{1}{4\alpha}\sqrt{\frac{t\beta}{\pi\alpha}}\left(3\beta-X\beta\sqrt{\alpha}+2t\alpha\right)\exp\left\{-\frac{1}{t}\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta}}\right)^2\right\}+\frac{t^2}{2}erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)$$

$$+\left(\frac{\beta}{4\alpha}\right)^2\left(3+2X\sqrt{\alpha}\right)\exp\left(-2X\sqrt{\alpha}\right)erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)+\left(\frac{\beta}{4\alpha}\right)^2\left(4X\sqrt{\alpha}-2X^2\alpha-3\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$J_6(X,t)=t\,erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)-\sqrt{\frac{t\beta}{\pi\alpha}}\exp\left\{-\frac{1}{t}\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta}}\right)^2\right\}+\frac{\beta}{4\alpha}\exp\left(-2X\sqrt{\alpha}\right)erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)$$

$$+\frac{\beta}{4\alpha}\left(2X\sqrt{\alpha}-1\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$J_7(X,t)=-\frac{1}{3\alpha}\sqrt{\frac{t\beta}{\pi\alpha}}\left(15\beta^2+6\beta^2X\sqrt{\alpha}+\beta^2\alpha X^2+4\alpha^2t^2+2\alpha\beta t\left(5+2X\sqrt{\alpha}\right)\right)\exp\left\{-\frac{1}{t}\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta}}\right)^2\right\}$$

$$+\frac{t}{3}\left(3\beta^2X^2+6\beta tX\sqrt{\alpha}+4t^2\alpha\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)+\frac{\beta^3}{12\alpha^2}\left(15+21X\sqrt{\alpha}+12\alpha X^2\right)$$

$$\exp\left(-2X\sqrt{\alpha}\right)erfc\left(\frac{\beta X-2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right)+\left(2X^3\alpha\sqrt{\alpha}-15+9X\sqrt{\alpha}\right)erfc\left(\frac{\beta X+2t\sqrt{\alpha}}{2\sqrt{\beta t}}\right),$$

$$\delta^2=\left(\alpha-\beta\gamma_0\right),\quad X=\frac{log(1+ax)}{a},\ U_0=u_0-aD_0,\ \gamma_0=au_0.$$

RESULTS AND DISCUSSIONS

The analytical solutions obtained as in Eq. (20), Eq.(21) and Eq. (22) are demonstrated with the help of input data to understand the solute concentration distribution behaviour in a finite domain $0\le x(\text{meter})\le 15$. The chosen sets of data are taken from the experimental and theoretical published literatures (Todd, 1980; Jaiswal et.al., 2009; Bharati et al., 2015; Singh et al., 2014). The domain is considered semi-infinite but the solute concentration increases with position in the time domain $0\le t<t_1$ (2 day) and decreases with position in the time domain $t\ge t_1$ (2 day) in a finite domain at different values of time.

To understand the concentration profiles, artificial data of continuous injection in the domain are used. Concentration values are evaluated from the Eqs. (20), (21) and Eq. (22) in a finite domain $0\le x(\text{meter})\le 15$ at different values of parameters such as time, dispersion coefficient and heterogeneous parameter. In this study the input parameters values and the ranges of these parameters in which they are varied taken either from published literature or empirical relationship. The concentration values C/C_0 are evaluated assuming the reference concentration as $C_0 = 1.0$, $C_i = 0.10$. The common input values are taken as $p = 0.01(\text{day}^{-2})$, $q = 0.02(\text{day}^{-1})$, $r = 0.03$, $t_1 = 2\,\text{day}$ and $t_2 = 5\,\text{day}$ for all cases. The medium is supposed heterogeneous. In this study source contamination is considered multiple instead of point source contamination.

Case-I: Figures (1-3) demonstrate the concentration behaviour in the time domain $(0 \le t < t_1 = 2\,\text{day})$ for the analytical solution obtained in Eq. (20).

Figure (1) illustrates the dimensionless concentration profiles at various time $t(\text{days}) = 0.5,\ 1.0$ and 1.5 with common parameters $u_0 = 1.10\,(\text{m/day})$, $D_0 = 2.18\,(\text{m}^2/\text{day})$, $a = 0.01(\text{m}^{-1})$. This figure exhibits that the input concentration that is the concentration at the origin of the domain are $0.033,\ 0.040,\ 0.054$ at time $t(\text{days}) = 0.5,\ 1.0$ and 1.5, respectively. Concentration level at the source boundary is higher for smaller time and lower for larger time. It attenuates with position and time. It's also clear that the rate of change in concentration on longitudinal direction is higher for lower time and attains a stationary position after a certain distance travelled onwards.

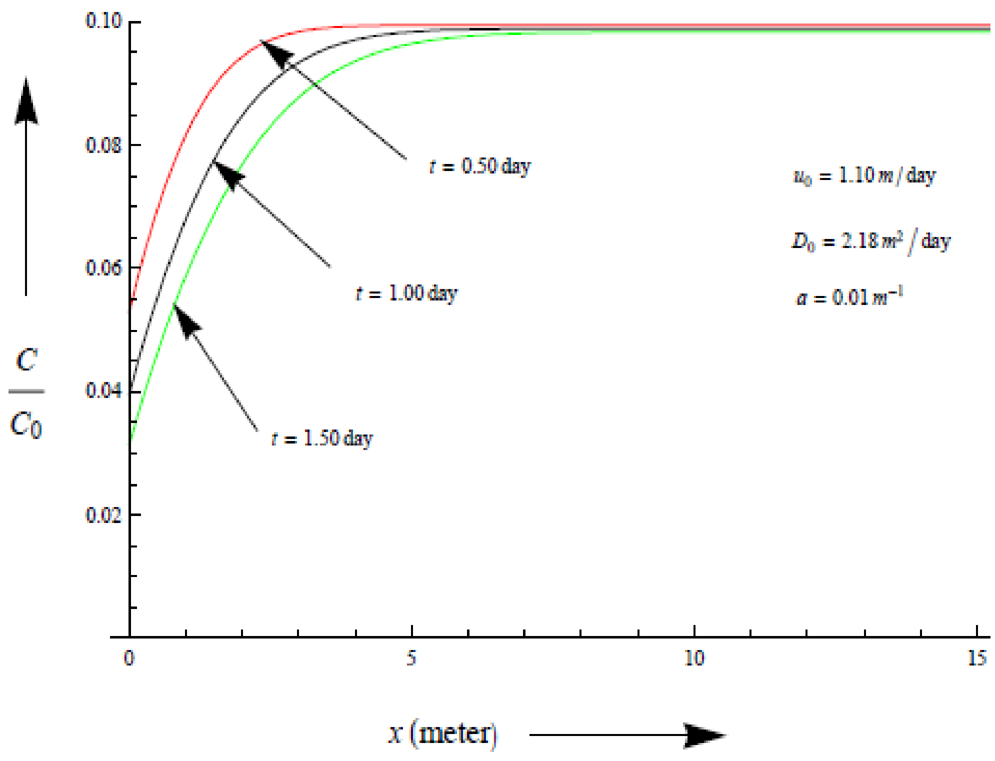

Fig. 1. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (20) at various time for $0 \le t < t_1$.

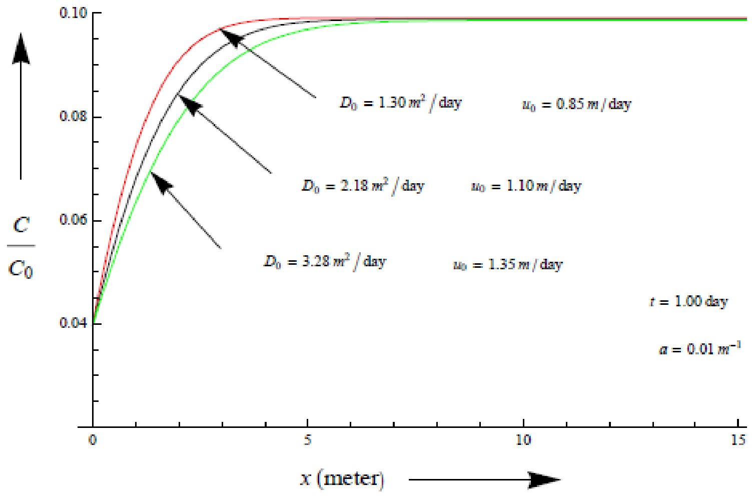

Fig. 2. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (20) at various dispersion parameter and velocity for $0 \le t < t_1$.

The contaminant concentration profile computed with various dispersion coefficient and corresponding seepage velocity $D_0 = 1.30\,(\mathrm{m}^2/\mathrm{day})$, $u_0 = 0.85\,(\mathrm{m/day})$, $D_0 = 2.18\,(\mathrm{m}^2/\mathrm{day})$, $u_0 = 1.10\,(\mathrm{m/day})$, and $D_0 = 3.28\,(\mathrm{m}^2/\mathrm{day})$, $u_0 = 1.35\,(\mathrm{m/day})$ with common parameters $t = 1.0\,(\mathrm{day})$ and $a = 0.01\,(\mathrm{m}^{-1})$ along longitudinal direction is shown in Figure (2). It is observed from that the input contaminant concentration on source boundary $x = 0$ is 0.040 for different dispersion coefficients. It attenuates with position and time. Enhance in the dispersion of the effluent would cause to its attenuation in the geological formation. The concentration pattern decreases to time, whereas increases to space and after a certain distance travelled it attains a stationary position.

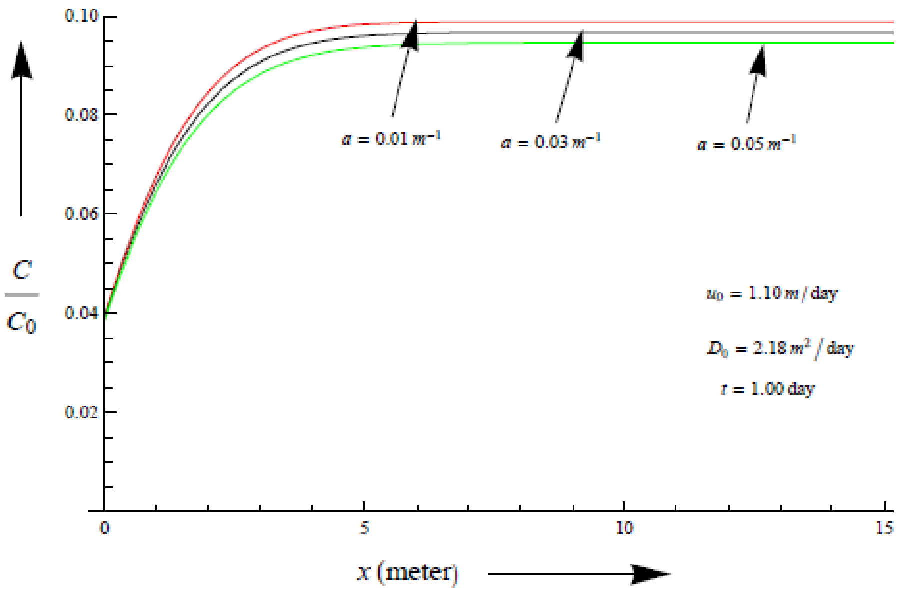

Fig. 3. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (20) at various heterogeneous parameters in time domain $0 \le t < t_1$.

Figure (3) demonstrate the dimensionless concentration distribution pattern computed at various heterogeneity parameters $a\,(\mathrm{m}^{-1}) = 0.01,\ 0.03,\ 0.05$ with common parameters $t = 1.0\,(\mathrm{day})$, $D_0 = 2.18\,(\mathrm{m}^2/\mathrm{day})$ and $u_0 = 1.10\,(\mathrm{m/day})$. It attenuates with position and time. At particular position the concentration level is lower for larger heterogeneous parameter and higher for the smaller heterogeneous parameter. The concentration pattern decreases with respect to heterogeneous parameter, whereas it increases with respect to the space and after a certain distance travelled it becomes constant for all time and space.

Case-II: Figures (4-6) demonstrate the concentration behaviour in the time domain $t_1\,(2\ \mathrm{day}) \le t < t_2\,(5\ \mathrm{day})$ for the analytical solution obtained in Eq. (21).

Figure (4) illustrated the dimensionless concentration distribution predicted by the present solution in Eq.(21) with different time $t\ (\mathrm{days}) = 3.5, 4.0$ and 4.5 computed for the common parameter $u_0 = 1.10\,(\mathrm{m/day})$, $D_0 = 2.18\,(\mathrm{m}^2/\mathrm{day})$ and $a = 0.01\,(\mathrm{m}^{-1})$. The input concentration C/C_0 at the origin ($x = 0$) are respectively 0.170, 0.235, 0.310 at the time $t\ (\mathrm{days}) = 3.5, 4.0$ and 4.5, respectively. It attenuates with position and time. At particular position the concentration level is lower for smaller time and higher for larger time. The concentration pattern decreases with respect to space and after a certain distance travelled it becomes constant for all time and space.

Figure (5) represents the dimensionless concentration distribution predicted by the present solution in Eq.(21) at various dispersion parameter and corresponding seepage velocity $D_0 = 1.30\,(\mathrm{m}^2/\mathrm{day})$, $u_0 = 0.85\,(\mathrm{m/day})$, $D_0 = 2.18\,(\mathrm{m}^2/\mathrm{day})$, $u_0 = 1.10\,(\mathrm{m/day})$, and $D_0 = 3.28\,(\mathrm{m}^2/\mathrm{day})$, $u_0 = 1.35\,(\mathrm{m/day})$ computed for the common parameter $t = 4.0\,(\mathrm{day})$, $a = 0.01\,(\mathrm{m}^{-1})$. It attenuates with position and time. At particular position the concentration level is lower for smaller dispersion parameter and higher for larger dispersion parameter. The concentration pattern decreases with respect to space and after a certain distance travelled it becomes constant for all time and space.

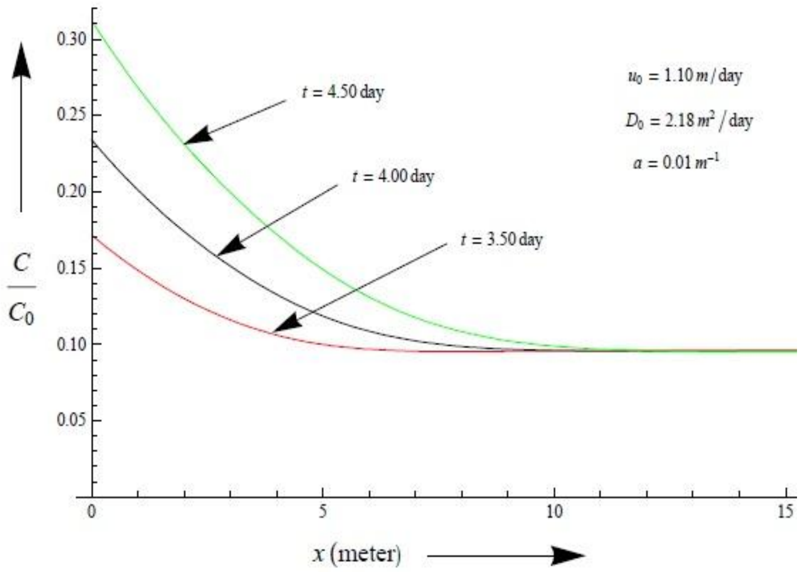

Fig. 4. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (21) at various time for $t_1 \le t < t_2$.

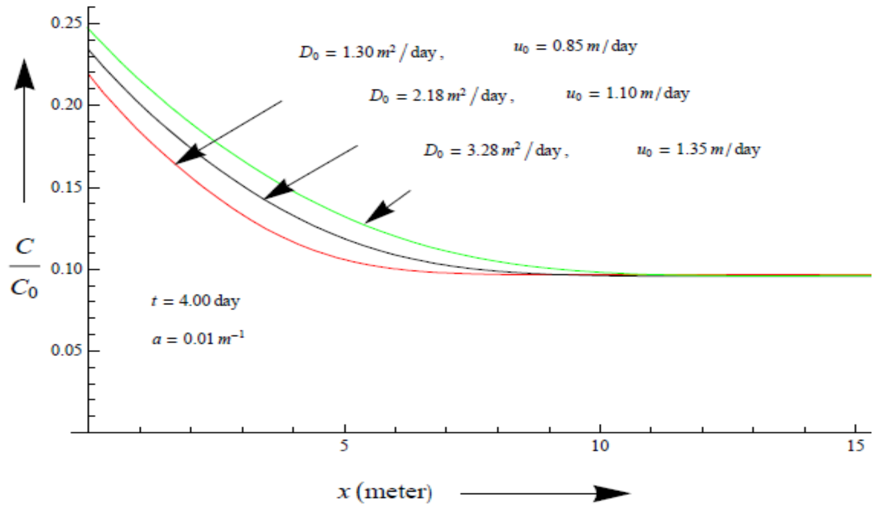

Fig. 5. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (21) at various dispersion parameter and velocity for $t_1 \le t < t_2$.

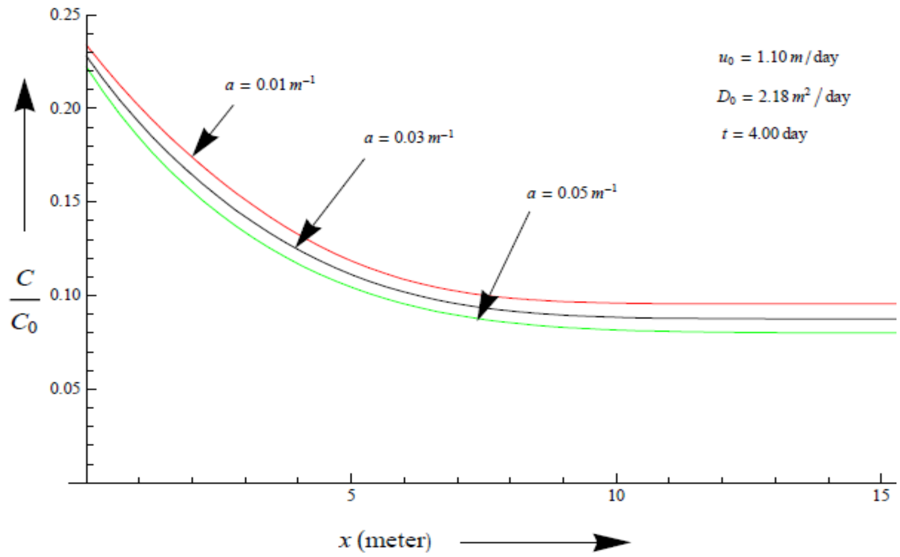

Fig. 6. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (21) for various heterogeneous parameter for $t_1 \le t < t_2$.

Figure (6) demonstrate the dimensionless concentration distribution pattern predicted by the present solution in Eq.(21) at various heterogeneous parameter $a\,(\text{m}^{-1}) = 0.01,\ 0.03,\ 0.05$, computed for the common parameter $t = 4.0\,(\text{day})$, $D_0 = 2.18(\text{m}^2/\text{day})$, $u_0 = 1.10\,(\text{m/day})$. It attenuates with position and time. At particular position the concentration level is lower for higher heterogeneous parameter and higher for the lower heterogeneous parameter. The concentration pattern decreases with respect to heterogeneous parameter and space but after certain distance travelled it becomes constant.

Case-III: Figures (7-9) demonstrate the concentration distribution in the time domain $t \ge t_2\,(5\,\text{day})$ for the analytical solution obtained in Eq. (22).

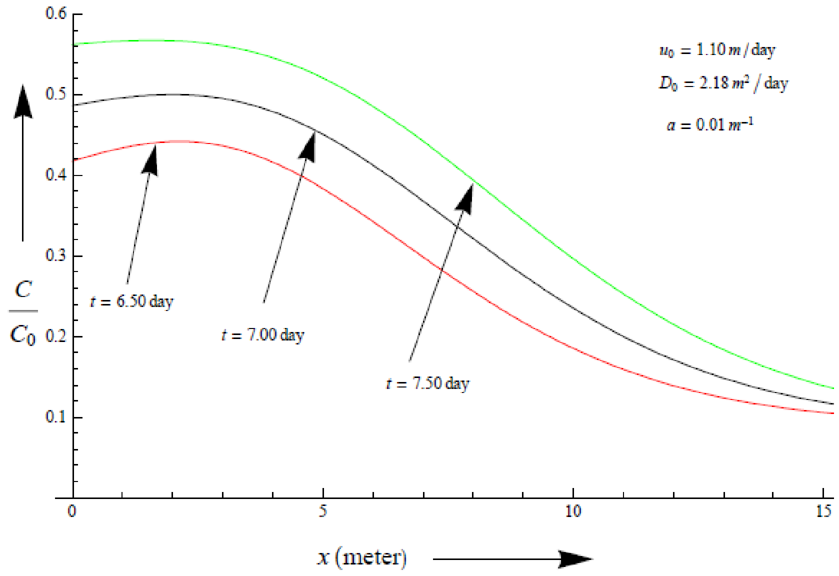

Fig. 7. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (22) at various time for $t \geq t_2$.

Figure (7) illustrated the dimensionless concentration distribution described by the analytical solution in Eq.(22) at different time $t\,(\text{days}) = 6.5, 7.0$ and 7.5 computed for the common parameter $u_0 = 1.10\,(\text{m/day})$, $D_0 = 2.18\,(\text{m}^2/\text{day})$, $a = 0.01\,(\text{m}^{-1})$. It attenuates with position and time. At particular position the concentration level is lower for smaller time and higher for larger time. The input concentration, C/C_0 at the origin ($x = 0$) are different at each time. The concentration pattern increases with respect to time and decreases with respect to space and after a certain distance travelled it becomes constant for all time and space.

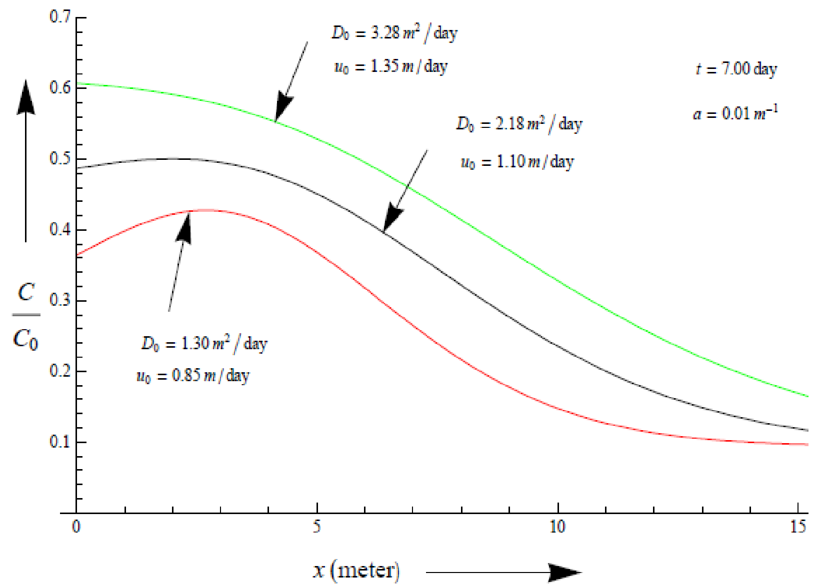

Fig. 8. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (22) at various dispersion parameter and velocity for $t \geq t_2$.

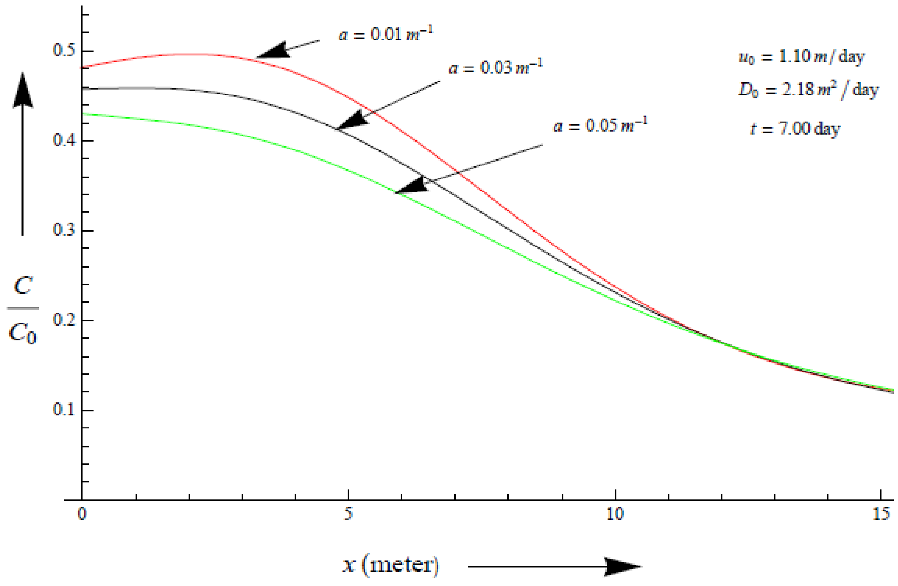

Fig. 9. Dimensionless concentration distribution evaluated by analytical solution presented in Eq. (22) for various heterogeneous parameter for $t \geq t_2$.

Figure (8) illustrates the solute transport from the point source along the longitudinal direction of the medium, presented in Eq.(22) at various dispersion coefficient and seepage velocity $D_0 = 1.30(\mathrm{m}^2/\mathrm{day})$, $u_0 = 0.85(\mathrm{m/day})$, $D_0 = 2.18(\mathrm{m}^2/\mathrm{day})$, $u_0 = 1.10(\mathrm{m/day})$ and $D_0 = 3.28(\mathrm{m}^2/\mathrm{day})$, $u_0 = 1.35(\mathrm{m/day})$ at time $t = 7.0(\mathrm{day})$ and $a = 0.01(\mathrm{m}^{-1})$. It attenuates with position and time. At particular position the concentration level is lower for smaller dispersion parameter and higher for a larger dispersion parameter. The concentration pattern decreases with space and after a certain distance it attains a stationary position.

Figure (9) illustrates the solute transport described by the solution in Eq.(22), in the time domain $t \geq t_2$ at various heterogeneity parameters $a(\mathrm{m}^{-1}) = 0.01,\ 0.03,\ \text{and}\ 0.05$, computed at $t = 7.0(\mathrm{day})$, $D_0 = 2.18(\mathrm{m}^2/\mathrm{day})$, $u_0 = 1.10(\mathrm{m/day})$. It attenuates with position and time. At particular position the concentration level is lower for larger heterogeneous parameter and higher for the smaller heterogeneous parameter. The concentration pattern decreases with respect to heterogeneous parameter and space, but after a certain distance travelled it becomes constant.

CONCLUSIONS

In this study, we studied analytical solutions to one-dimensional advection dispersion equation for conservative solute transport with several point source boundary conditions. The geological formation of the domain is considered semi-infinite and heterogeneous in nature. The dispersion coefficient is assumed to vary as a square function of distance. The solutions are obtained by using the Laplace transform technique. In Laplace transformation technique the solution is obtained by transforming the advection dispersion equation into an ordinary differential equation with help of certain other transformation. The solutions to all possible combinations of spatially dependence are demonstrated with the help of graphs. The developed analytical solutions may help as a useful tool for evaluating the aquifer concentration at any position and time. Such solutions are useful in validating a numerical solution to a dispersion problem. Derived solution can be extended for any time-dependent boundary conditions. The analytical model presented here provides better information about various physical transport parameters.

ACKNOWLEDGEMENT

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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