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Solute Transport for Pulse Type Input Point Source along Temporally and Spatially Dependent Flow

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ABSTRACT: In the present study, analytical solutions are obtained for two-dimensional advection dispersion equation for conservative solute transport in a semi-infinite heterogeneous porous medium with pulse type input point source of uniform nature. The change in dispersion parameter due to heterogeneity is considered as linear multiple of spatially dependent function and seepage velocity whereas seepage velocity is n^{th} power of spatially dependent function. Two forms of the seepage velocity namely exponentially decreasing and sinusoidal form are considered. First order decay and zero order production are also considered. The geological formation of the porous medium is considered of heterogeneous and adsorbing nature. Domain of the medium is uniformly polluted initially. Concentration gradient is considered zero at infinity. Certain new transformations are introduced to transform the variable coefficients of the advection diffusion equation into constant coefficients. Laplace Transform Technique (LTT) is used to obtain analytical solutions of advection-diffusion equation. The solutions in all possible combinations of temporally and spatially dependence dispersion are demonstrated with the help of graphs.

Keywords: Advection, Dispersion, Retardation factor, Point source, Heterogeneous medium.

INTRODUCTION

The development of mathematical models plays an important role in understanding and prediction of solute transport phenomenon in an aquifer. Prediction of contaminant transport in porous media is a critical prerequisite for waste containment blueprint and evaluation of remediation effort. The rate of contaminant attenuation in subsurface depends on mixing process caused by dispersion which is mainly occurs due to special variation in aquifer properties like, variation in hydraulic conductivity. Solute transport in porous media is mathematically governed by advection dispersion equation which is a second order parabolic partial differential equation based upon the conservation of mass and Fick's first law of diffusion. A number of analytical solutions describing groundwater flow and solute transport in porous media have been published in literature. Notable analytical solutions for advection-dispersion equation are in favor of (Banks & Ali, 1964; Ogata, 1970; Marino, 1974; Al-Niami & Rushton, 1977) considering unsteady seepage velocity, constant dispersion, adsorption, first order decay and zero order production, etc. Yates (1990 & 1992) obtained the analytical solutions for linearly or exponentially increasing dispersion coefficient in one-dimensional porous media. Aral & Liao (1996) obtained analytical solutions of two-

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dimensional advection dispersion equation with a time-dependent dispersion coefficient. Hunt (1998) discussed analytical solutions for an instantaneous source and for steady flow with a continuous source in one, two and three-dimensional advection dispersion equation with scale-dependent dispersion coefficients. Chen et al. (2003, 2008) obtained the analytical solutions in a cylindrical coordinate system with distance-dependent dispersion of a tracer in convergent and divergent radially symmetric flow fields. Su et al. (2005) obtained similarity solution using a time and scale-dependent dispersivity in fractal porous media. Smedt (2006) presented analytical solutions for solute transport in rivers considering the effects of first order decay and transient storage. Zhan et al. (2009) obtained an analytical solution for two-dimensional solute transport using first and third type boundary conditions. Chen and Liu (2011) developed analytical solutions for advection-dispersion equation in finite domain with arbitrary time-dependent boundary conditions. Yadav and Jaiswal (2011) obtained an analytical solution of temporally dependent solute dispersion in a two-dimensional shallow aquifer while longitudinal solute transport for a pulse type source along temporally and spatially dependent flow was discussed by Yadav et al. (2012). Singh et al. (2013) discussed analytical solutions for time-dependent point-source in two-dimensional homogeneous porous medium. Bing et al. (2015) discussed analytical solutions for solute transport in one-dimension semi-infinite porous media using the source function method. Most of these works have included the attenuation effect due to adsorption, first order decay and/or chemical reactions. Majdalani et al. (2015) obtained analytical solution of solute transport with scale dependent dispersion in a heterogeneous porous media. Sanskrityayn et al. (2016) developed analytical solution of advection-dispersion equation with space and time dependent dispersion using Green's function. Djordjevich & Savovic (2017) developed numerical solution for two-dimensional solute transport with periodic flow in homogeneous porous media using finite difference technique. Das et al. (2018) discussed analytical and numerical solutions for solute transport modelling in homogeneous semi-infinite porous medium with the variable temporally dependent boundary. Yadav and Kumar (2018) developed a mathematical model for two-dimensional solute transport in a semi-infinite heterogeneous porous medium with spatially and temporally dependent coefficients for pulse type input concentration of varying nature. Sanskrityayn et al. (2018) obtained analytical solution of solute transport due to spatio-temporally dependent dispersion coefficient and velocity in a heterogeneous porous medium. Two-dimensional contaminant transport models have multiple advantages over one-dimensional models. For example, two-dimensional models can account for concentration gradients and contaminant transport in the direction perpendicular to the groundwater flow. In previous published literature almost all solutions derived in two-dimensions in which only longitudinal velocity component were considered, neglecting transverse velocity component, while in the present study longitudinal and lateral directions of dispersion coefficients and velocity components are considered. The dispersion parameter is considered as linear multiple of spatially dependent function and seepage velocity while the seepage velocity is the n^{th} power of spatially dependent function. Two form of ground water velocity namely sinusoidal form and exponential decreasing form of time varying function are taken into account. The concentration of the inlet stream is not zero it means the chamber is fed with polluted water. The pulse type conservative solute is introduced at the origin of the domain and other end considered of flux type boundary condition. The first order decay and zero order production term are also considered. Geological formations are taken semi-infinite and adsorbing nature. The medium is considered heterogeneous as a result the velocity of the flow field is considered a spatially dependent function in both the directions. Analytical solution is obtained with the help of Laplace Transformation Technique for uniform input point source concentration. Concentration distributions are demonstrated graphically.

THEORY AND METHODS

The pollutants are entered in subsurface by mainly two mechanism first one, advection which is caused by flow of groundwater and second one, by dispersion which is caused by mechanical mixing and molecular diffusion. Molecular diffusions are not taken into consideration due to small seepage velocity. The mathematical form of the advection-diffusion equation in two-dimensions can be given by a second order partial differential equation of parabolic type which is written as:

$$R\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(D_x\frac{\partial C}{\partial x}-uC\right)+\frac{\partial}{\partial y}\left(D_y\frac{\partial C}{\partial y}-vC\right)-\gamma C+\mu \qquad (1)$$

In which $C[\mathrm{ML}^{-3}]$ is the solute concentration of the pollutant transporting along the flow field through the medium at any position (x, y) and time t. $D_x[\mathrm{L}^2\mathrm{T}^{-1}]$ and $D_y[\mathrm{L}^2\mathrm{T}^{-1}]$ are the longitudinal and transverse dispersion coefficient respectively while $u[\mathrm{LT}^{-1}]$ and $v[\mathrm{LT}^{-1}]$ are the unsteady uniform seepage velocity along longitudinal and transverse directions respectively. $\gamma[\mathrm{T}^{-1}]$ is the first order decay constant and $\mu[\mathrm{ML}^{-3}\mathrm{T}^{-1}]$ is the zero order production rate coefficient for solute which represents internal/external production of the solute in the medium. First term on the left hand side of the Eq.(1) is represent change in concentration with time in liquid phase and R is the retardation factor which is a dimensionless quantity. First term on the right-hand side of the Eq.(1) describes the influence of the dispersion on the concentration distribution in longitudinal direction while second term is the change of concentration due to advective transport in longitudinal direction. Third term on the right-hand side of the Equation (1) describes the influence of the dispersion on the concentration distribution in transverse direction while fourth term is the change of concentration due to advective transport in transverse direction. Fifth and sixth term on the right-hand side of the Eq.(1) describe the first order decay and zero order production respectively.

The medium through which the solute dispersion occurs is supposed to be of semi-infinite extended along the longitudinal $0\le x(m)\le 5$ and transverse $0\le y(m)\le 3$ directions. Let the ground water velocity components of the flow satisfy the Darcy's law in both directions. Let pollutant enter in the medium continuously at a uniform rate up to a certain time period and just after, it becomes zero. In other words, the source of pollution is invariably uniform pulse-type point source. A set of initial and boundary conditions are assumed to solve the advection-dispersion equation. Initially the semi-infinite medium is considered not solute free. Let the medium be horizontal and solute particle entered from the origin. Let t_0 be the time of elimination of the point source. Flux type homogeneous conditions are assumed at far ends of the medium, along both the directions. Mathematically initial and boundary conditions may be written as:

$$C(x,y,t)=C_i \;\; ; \;\; t=0\;, x\ge 0, y\ge 0 \qquad (2)$$

$$C(x,y,t)=\begin{cases} C_0 & ; \; 0<t\le t_0 \\ & \qquad ; \; x=0, y=0 \\ 0 & ; \; t>t_0 \end{cases} \qquad (3)$$

$$\frac{\partial C(x,y,t)}{\partial x}=0, \quad \frac{\partial C(x,y,t)}{\partial y}=0; \qquad (4)$$
$$t\ge 0, x\to\infty, y\to\infty$$

Equation (2) represents initially there is some concentration present in the domain. C_i is the initial concentration present in the domain. A pulse-type input condition represented by Equation (3), in which t_0 is the time span of the contaminant release (assuming release starts at time zero) and C_0 is the constant concentration of the pulse at the inlet boundary. Equation (4) indicates that concentration gradient is zero at infinity. The medium is considered heterogeneous. The temporal variation in velocity is the result of temporal variation in the hydraulic gradient. Sykes et al., (1982) and Sudicky, (1986) have concluded that the magnitude and direction of the spatial mean hydraulic gradient fluctuate over time. In natural flow systems these are rarely in a steady state. Dispersion coefficient and velocity both are considered spatially and temporally dependent in general form. Some particular expressions are chosen. In the present study retardation factor is also considered in degenerate form. Thus the expressions for velocity and dispersion coefficient components are written in the general form as:

$$\left.\begin{aligned} &u = u_0\left(1+ax\right)^n\left(1+by\right)^{n-1} f\left(mt\right),\\ &v = v_0\left(1+by\right)^n\left(1+ax\right)^{n-1} f\left(mt\right),\\ &D_x = D_{x_0}\left(1+ax\right)^{n+1}\left(1+by\right)^{n-1} f\left(mt\right),\\ &D_y = D_{y_0}\left(1+by\right)^{n+1}\left(1+ax\right)^{n-1} f\left(mt\right),\\ &\gamma = \gamma_0\left(1+ax\right)^{n-1}\left(1+by\right)^{n-1} f\left(mt\right),\\ &\mu = \mu_0\left(1+ax\right)^{n-1}\left(1+by\right)^{n-1} f\left(mt\right),\\ &R = R_0\left(1+ax\right)^{n-1}\left(1+by\right)^{n-1},\\ &\qquad where\ \ n \in I\left(Set\, of\ integers\right) \end{aligned}\right\} \tag{5}$$

where a, b are the heterogeneity parameters along longitudinal and lateral directions, respectively, have dimension inverse of space variable (Kumar et al., 2010). The various value of (a, b) represents different heterogeneity. Heterogeneity of the porous medium means porosity or hydraulic conductivity is dependent upon position. D_{x_0}, D_{y_0}, u_0 and v_0 are initial dispersion coefficients and unsteady uniform seepage velocities along longitudinal and transverse directions respectively. γ_0, μ_0 and R_0 are the initial first order decay, zero order production and retardation factor, respectively. m represents unsteady parameter whose dimension is inverse of time variable t. $m = 0$ corresponds to the temporally independent parameters. It is assumed that $f(mt) = 1$ for $m = 0$ or $t = 0$. The first case represents the steady flow and second case represents the initial state. Thus $f(mt)$ is a non-dimensional expression. In the proposed problem two form of $f(mt)$ namely sinusoidal form of time varying $f(mt) = 1 - \sin(mt)$ and exponential decreasing form of time varying $f(mt) = \exp(-mt)$ are taken.

Substituting values from Equation (5) in Equation (1), we have

$$\begin{aligned} R_0\frac{\partial C}{\partial t}\frac{1}{f(mt)} &= D_{x_0}(1+a\,x)^2\frac{\partial^2 C}{\partial x^2} + D_{y_0}(1+b\,y)^2\frac{\partial^2 C}{\partial y^2} - \{u_0 - a\ (n+1)D_{x_0}\}\ (1+a\,x)\frac{\partial C}{\partial x}\\ &\quad - \{v_0 - b\ (n+1)\,D_{y_0}\}\ (1+b\,y)\frac{\partial C}{\partial y} - n\,a\,u_0\,C - n\,b\,v_0\,C - \gamma_0\,C + \mu_0 \end{aligned} \tag{6}$$

Let us introduce new independent space variables X and Y defined as (Kumar et al., 2010);

$$\left.\begin{aligned} X &= \frac{log\left(1+ax\right)}{a} \Rightarrow \frac{dX}{dx} = \frac{1}{\left(1+ax\right)}\\ y &= \frac{log\left(1+by\right)}{b} \Rightarrow \frac{dY}{dy} = \frac{1}{\left(1+by\right)} \end{aligned}\right\} \tag{7}$$

With the help of these transformations, Equation (6) can be written as:

$$\begin{aligned} &R_0\frac{\partial C}{\partial t}\frac{1}{f\left(mt\right)} = D_{x_0}\frac{\partial^2 C}{\partial X^2} + D_{y_0}\frac{\partial^2 C}{\partial Y^2} -\\ &\left(u_0 - naD_{x_0}\right)\frac{\partial C}{\partial X} - \left(v_0 - nbD_{y_0}\right)\frac{\partial C}{\partial Y} -\\ &nau_0C - nbv_0C - \gamma_0 C + \mu_0 \end{aligned} \tag{8}$$

Equations (2-4) may be written in terms of new independent space variables, X and Y as follows:

$$C(X,Y,t)=C_i \quad ; \quad t=0, X\geq 0, Y\geq 0 \tag{9}$$

$$C(X,Y,t)=\begin{cases} C_0 & ; \quad 0<t\leq t_0 \\ 0 & ; \quad t>t_0 \end{cases}, \quad X=0, Y=0 \tag{10}$$

$$\frac{\partial C(X,Y,t)}{\partial X}=0, \frac{\partial C(X,Y,t)}{\partial Y}=0 \quad ; \; t\geq 0, \; X\to\infty, Y\to\infty \tag{11}$$

Let a new independent space variable, Z be introduced as (Carnahan & Remer, 1984):

$$Z=X+Y \tag{12}$$

Using this transformation, Equation (8) may be written as:

$$R_0\frac{\partial C}{\partial t}\frac{1}{f(mt)}=D_0\frac{\partial^2 C}{\partial Z^2}-U_0\frac{\partial C}{\partial Z}-\gamma_1 C+\mu_0 \tag{13}$$

where

$D_0=D_{x_0}+D_{y_0}$, $U_0=u_0+v_0-n\left(aD_{x_0}+bD_{y_0}\right)$, $\gamma_1=\gamma_0+n\left(au_0+bv_0\right)$.

Equation (9-11) may be written in terms of new independent space variable Z as:

$$C(Z,t)=C_i \quad ; \quad t=0, Z\geq 0 \tag{14}$$

$$C(Z,t)=\begin{cases} C_0 & ; \quad 0<t\leq t_0 \\ 0 & ; \quad t>t_0 \end{cases}, \; Z=0 \tag{15}$$

$$\frac{\partial C(Z,t)}{\partial Z}=0 \quad ; \; t\geq 0, \quad Z\to\infty \tag{16}$$

Let us introduce a new time variable, T by the following transformation (Crank, 1975):

$$T=\int_0^t f(mt)\,dt \tag{17}$$

where $f(mt)$ is taken as two form of time varying function, namely exponential decreasing and sinusoidal. Therefore from this transformation Equation (17), we have

$$T=\frac{1}{m}\{1-\exp(-mt)\} \quad \text{for} \quad f(mt)=\exp(-mt) \tag{17.a}$$

$$T=\frac{1}{m}\left[mt-\{1-\cos(mt)\}\right] \quad \text{for} \quad f(mt)=1-\sin(mt) \tag{17.b}$$

Using transformation Equation (17) in Equation (13), it reduces into

$$R_0\frac{\partial C}{\partial T}=D_0\frac{\partial^2 C}{\partial Z^2}-U_0\frac{\partial C}{\partial Z}-\gamma_1 C+\mu_0 \tag{18}$$

Equations (14-16) may be written in terms of new time variable T as:

$$C(Z,T)=C_i \quad ; \quad T=0, \; Z\geq 0 \tag{19}$$

$$C(Z,T)=\begin{cases} C_0 & ; \quad 0<T\leq T_0 \\ 0 & ; \quad T>T_0 \end{cases}, \; Z=0 \tag{20}$$

$$\frac{\partial C(Z,t)}{\partial Z}=0 \quad ; \; T\geq 0, \quad Z\to\infty \tag{21}$$

Now we take another transformation

$$C(Z,T)=K(Z,T)\exp\left\{\frac{U_0 Z}{2D_0}-\frac{T}{R_0}\left(\frac{U_0^2}{4D_0}+\gamma_1\right)\right\}+\frac{\mu_0}{\gamma_1} \tag{22}$$

Equations (18-21) becomes

$$R_0\frac{\partial K}{\partial T}=D_0\frac{\partial^2 K}{\partial Z^2} \tag{23}$$

$$K(Z,T)=\left(C_i-\frac{\mu_0}{\gamma_1}\right)\exp\left(-\frac{U_0}{2D_0}Z\right) \quad ; T=0, Z\geq 0 \tag{24}$$

$$K(Z,T)=\begin{cases} \left(C_0-\frac{\mu_0}{\gamma_1}\right)\exp\left(\alpha^2 T\right) & ; \; 0<T\leq T_0 \\ -\frac{\mu_0}{\gamma_1}\exp\left(\alpha^2 T\right) & ; \; T>T_0 \end{cases}, \; Z=0 \tag{25}$$

$$\frac{\partial K(Z,T)}{\partial Z}+\frac{U_0}{2D_0}K=0 \quad ; \; T\geq 0, \; Z\to\infty \tag{26}$$

where

$$\alpha^2=\frac{1}{R_0}\left(\frac{U_0^2}{4D_0}+\gamma_1\right)$$

Applying the Laplace Transformation on Equations (23-26) , one can find transformed equation and boundary condition as:

$$\frac{d^2\bar{K}}{dZ^2}-\frac{pR_0}{D_0}\bar{K}=-\frac{R_0}{D_0}\left(C_i-\frac{\mu_0}{\gamma_1}\right)\exp\left(-\frac{U_0}{2D_0}Z\right) \tag{27}$$

$$\bar{K}(Z,p)=\frac{C_0}{p-\alpha^2}\left[1-\exp\left\{-\left(p-\alpha^2\right)T_0\right\}\right]-\frac{\mu_0}{\gamma_1\left(p-\alpha^2\right)}; \quad Z=0 \tag{28}$$

$$\frac{d\bar{K}}{dZ}+\frac{U_0}{2D_0}\bar{K}=0 \quad ; \quad Z\to\infty \tag{29}$$

where $\overline{K} = \int_0^{\infty} K(Z,T) e^{-pT} dT$ and p Laplace parameter.

Now the general solution of Equation (27) may be written as:

$$\overline{K}(Z,p) = c_1 \exp\left(-Z\sqrt{\frac{pR_0}{D_0}}\right) + c_2 \exp\left(Z\sqrt{\frac{pR_0}{D_0}}\right) + \left(C_i - \frac{\mu_0}{\gamma_1}\right)\frac{\exp\left(-U_0 Z/2D_0\right)}{(p-\beta^2)} \tag{30}$$

where $\beta^2 = \dfrac{U_0^2}{4R_0 D_0}$ and c_1 & c_2 are arbitrary constants.

With the help of Equation (28) and Equation (29), one can get particular solution of Equation (30) as:

$$\begin{aligned}\overline{K}(Z,p) = {} & \frac{C_0}{p-\alpha^2}\left[1-\exp\{-(p-\alpha^2)T_0\}\right]\exp\left(-Z\sqrt{\frac{pR_0}{D_0}}\right) - \frac{\mu_0}{\gamma_1(p-\alpha^2)}\exp\left(-Z\sqrt{pR_0/D_0}\right) \\ & -\left(C_i - \frac{\mu_0}{\gamma_1}\right)\frac{\exp\left(-Z\sqrt{pR_0/D_0}\right)}{(p-\beta^2)} + \left(C_i - \frac{\mu_0}{\gamma_1}\right)\frac{\exp\left(-U_0 Z/2D_0\right)}{(p-\beta^2)}\end{aligned} \tag{31}$$

Now, Applying inverse Laplace Transformation and back transformations Equation (22), Equation (17), Equation (12) and Equation (7) as (Van Genuchten & Alves, 1982),

$$C(Z,T) = \frac{\mu_0}{\gamma_1} + \left(C_0 - \frac{\mu_0}{\gamma_1}\right)F(Z,T) + \left(C_i - \frac{\mu_0}{\gamma_1}\right)G(Z,T) \quad ; \quad 0 < T \le T_0 \tag{32.a}$$

$$C(Z,T) = \frac{\mu_0}{\gamma_1} + \left(C_0 - \frac{\mu_0}{\gamma_1}\right)F(Z,T) - C_0 F(Z,T-T_0) + \left(C_i - \frac{\mu_0}{\gamma_1}\right)G(Z,T) \quad ; \quad T > T_0 \tag{32.b}$$

where

$$\begin{aligned}F(Z,T) = {} & \frac{1}{2}\exp\left[\frac{\left\{U_0 - \left(U_0^2 + 4\gamma_1 D_0\right)^{1/2}\right\}Z}{2D_0}\right] erfc\left[\frac{R_0 Z - \left(U_0^2 + 4\gamma_1 D_0\right)^{1/2} T}{2\sqrt{D_0 R_0 T}}\right] \\ & + \frac{1}{2}\exp\left[\frac{\left\{U_0 + \left(U_0^2 + 4\gamma_1 D_0\right)^{1/2}\right\}Z}{2D_0}\right] erfc\left[\frac{R_0 Z + \left(U_0^2 + 4\gamma_1 D_0\right)^{1/2} T}{2\sqrt{D_0 R_0 T}}\right]\end{aligned}$$

$$G(Z,T) = \exp\left(-\frac{\gamma_1}{R_0}T\right)\left\{1 - \frac{1}{2}erfc\left(\frac{R_0 Z - U_0 T}{2\sqrt{D_0 R_0 T}}\right) - \frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{R_0 Z + U_0 T}{2\sqrt{D_0 R_0 T}}\right)\right\}$$

$$Z = \frac{\log(1+ax)}{a} + \frac{\log(1+by)}{b}, \quad T = \int_0^t f(mt)\,dt, \; D_0 = D_{x_0} + D_{y_0}, U_0 = u_0 + v_0 - n(aD_{x_0} + bD_{y_0}),$$

$$\gamma_1 = \gamma_0 + n(au_0 + bv_0).$$

Solution obtained in Equation (32.a) represents the solute concentration in the presence of source, in the time domain $t \le t_0$ beyond this time the concentration values are evaluated from the solution obtained in Equation (32.b). The obtained solutions have several application and extension. Some known solutions are derived as particular case from the obtained solution of the present study. It accomplishes the validation of the mathematical formulation and analytical procedure obtaining the solution.

If we put $\mu = 0, a = 0, b = 0$ and $m = 0$, in the Equation (32a, b) it has good agreement to the result obtained by Yadav et al. (2011) for two dimensional steady flow solute transport and may be given as:

$$C(Z,t) = C_0\,F(Z,t) + C_i\,G(Z,t) \quad ; \quad 0 < t \le t_0 \tag{33.a}$$

$$C(Z,t) = C_0 F(Z,t) - C_0\,F(Z,t-t_0) + C_i\,G(Z,t) \quad ; \quad t > t_0 \tag{33.b}$$

where

$$F(Z,t) = \frac{1}{2}\exp\left[\frac{\left\{U_0 - \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\right\}Z}{2D_0}\right] erfc\left[\frac{R_0\,Z - \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\,t}{2\sqrt{D_0\,R_0 t}}\right]$$
$$+\frac{1}{2}\exp\left[\frac{\left\{U_0 + \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\right\}Z}{2\,D_0}\right] erfc\left[\frac{R_0\,Z + \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\,t}{2\sqrt{D_0\,R_0 t}}\right]$$

$$G(Z,t) = \exp\left(-\frac{\gamma_0}{R_0}t\right)\left\{1 - \frac{1}{2}erfc\left(\frac{R_0\,Z - U_0\,t}{2\sqrt{D_0 R_0 t}}\right) - \frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right) erfc\left(\frac{R_0\,Z + U_0\,t}{2\sqrt{D_0 R_0 t}}\right)\right\}$$

$Z = x + y, D_0 = D_{x_0} + D_{y_0}, U_0 = u_0 + v_0$.

If we extend in $\mu = 0, y = 0, a = 0$ and $f(mt) = \exp(-mt)$ in Equation (32a,b) it shows good agreement with result obtained by Yadav et al. (2011) for one-dimensional unsteady flow and may be written as:

$$C(x,T) = C_0 F(x,T) + C_i G(x,T) \quad ; \quad 0 < T \le T_0 \tag{34.a}$$

$$C(x,T) = C_0 F(x,T) - C_0 F(x,T-T_0) + C_i G(x,T) \quad ; \quad T > T_0 \tag{34.b}$$

where

$$F(x,T) = \frac{1}{2}\exp\left[\frac{\left\{U_0 - \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\right\}x}{2\,D_0}\right] erfc\left[\frac{R_0\,x - \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\,T}{2\sqrt{D_0\,R_0 T}}\right]$$
$$+\frac{1}{2}\exp\left[\frac{\left\{U_0 + \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\right\}x}{2\,D_0}\right] erfc\left[\frac{R_0\,x + \left(U_0^2 + 4\gamma_0\,D_0\right)^{1/2}\,T}{2\sqrt{D_0\,R_0 T}}\right]$$

$$G(x,T) = \exp\left(-\frac{\gamma_0}{R_0}T\right)\left\{1 - \frac{1}{2}erfc\left(\frac{R_0 x - U_0 T}{2\sqrt{D_0 R_0 T}}\right) - \frac{1}{2}\exp\left(\frac{xU_0}{D_0}\right) erfc\left(\frac{R_0 x + U_0 T}{2\sqrt{D_0 R_0 T}}\right)\right\}$$

$T = \frac{1}{m}\{1 - \exp(-mt)\}$, $D_0 = D_{x_0}$, $U_0 = u_0$.

If we put $\mu = 0, a = 0, b = 0$ and $f(mt) = \exp(-mt)$, in the Equation (32a,b), the obtained result again fully matched with the result derived by Yadav et al. (2011) for two-dimensional unsteady flow and may be written as:

$$C(Z,T) = C_0 F(Z,T) + C_i G(Z,T) \quad ; \quad 0 < T \le T_0 \tag{35.a}$$

$$C(Z,T) = C_0\,F(Z,T) - C_0\,F(Z,T-T_0) + C_i\,G(Z,T) \quad ; \quad T > T_0 \tag{35.b}$$

where

$$F(Z,T)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_0 D_0\right)^{1/2}\right\}Z}{2D_0}\right]erfc\left[\frac{R_0Z-\left(U_0^2+4\gamma_0 D_0\right)^{1/2}T}{2\sqrt{D_0R_0T}}\right]$$

$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_0 D_0\right)^{1/2}\right\}Z}{2D_0}\right]erfc\left[\frac{R_0Z+\left(U_0^2+4\gamma_0 D_0\right)^{1/2}T}{2\sqrt{D_0R_0T}}\right]$$

$$G(Z,T)=\exp\left(-\frac{\gamma_0}{R_0}T\right)\left\{1-\frac{1}{2}erfc\left(\frac{R_0Z-U_0T}{2\sqrt{D_0R_0T}}\right)-\frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{R_0Z+U_0T}{2\sqrt{D_0R_0T}}\right)\right\}$$

$$Z=x+y,\ T=\frac{1}{m}\{1-\exp(-mt)\},\ D_0=D_{x_0}+D_{y_0},\ U_0=u_0+v_0.$$

If we put parameters and variable $n=-1$, $y=0$, $m=0$ and $\gamma=0$, $\mu=0$, $C_i=0$, in the Equation (32a, b) the obtained result again fully matched with the result derived by Kumar & Yadav (2015) for one-dimensional steady flow and may be given as:

$$C(X,t)=C_0\,F(X,t)\ \ ;\ \ 0<t\le t_0 \tag{36.a}$$

$$C(X,t)=C_0\,F(X,t)-C_0\,F(X,t-t_0)\ \ ;\ \ t>t_0 \tag{36.b}$$

where

$$F(X,t)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}X}{2D_0}\right]erfc\left[\frac{R_0X-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}t}{2\sqrt{D_0R_0t}}\right]$$

$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}X}{2D_0}\right]erfc\left[\frac{R_0X+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}t}{2\sqrt{D_0R_0t}}\right]$$

$$X=\frac{\log(1+a\,x)}{a},\ D_0=D_{x_0},\ U_0=u_0+a\,D_{x_0},\ \ \gamma_1=\gamma_0-a\,u_0$$

If we put the parameters and variable $n=1$, $m=0$, $R=1$, $\gamma=0$ and $\mu=0$, in the Equation (32a,b), it shows good agreement of the result obtained by Jaiswal et al. (2011) for two-dimensional steady flow solution and may be given as:

$$C(Z,t)=C_0\,F(Z,t)+C_i\,G(Z,t)\ \ ;\ \ 0<t\le t_0 \tag{37.a}$$

$$C(Z,t)=C_0\,F(Z,t)-C_0\,F(Z,t-t_0)+C_i\,G(Z,t)\ ;\ t>t_0 \tag{37.b}$$

where

$$F(Z,t)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}Z}{2D_0}\right]erfc\left[\frac{Z-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}t}{2\sqrt{D_0t}}\right]$$

$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}Z}{2D_0}\right]erfc\left[\frac{Z+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}t}{2\sqrt{D_0t}}\right]$$

$$G(Z,t)=\exp(-\gamma_1 t)\left\{1-\frac{1}{2}erfc\left(\frac{Z-U_0\,t}{2\sqrt{D_0t}}\right)-\frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{Z+U_0\,t}{2\sqrt{D_0t}}\right)\right\}$$

$$Z=\frac{\log(1+ax)}{a}+\frac{\log(1+by)}{b},\ D_0=D_{x_0}+D_{y_0},U_0=u_0+v_0-(aD_{x_0}+bD_{y_0}),\quad \gamma_1=(au_0+bv_0)$$

If we put $n=1,y=0,m=0$ and $R=1,\gamma=0,\mu=0,C_i=0$, in the Equation (32a,b) it shows good agreement with the result obtained by Kumar et al. (2010) for one-dimensional steady flow solution with continuous input concentration of uniform nature and may be written as:

$$C(X,t)=C_0F(X,t) \tag{38}$$

where

$$F(X,t)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_1D_0\right)^{1/2}\right\}X}{2\ D_0}\right]erfc\left[\frac{X-\left(U_0^2+4\gamma_1D_0\right)^{1/2}\ t}{2\sqrt{D_0\,t}}\right]$$
$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_1D_0\right)^{1/2}\right\}X}{2\ D_0}\right]erfc\left[\frac{X+\left(U_0^2+4\gamma_1D_0\right)^{1/2}\ t}{2\sqrt{D_0\,t}}\right]$$

$$X=\frac{\log(1+ax)}{a},\ D_0=D_{x_0},\ U_0=u_0-aD_{x_0},\ \gamma_1=au_0.$$

If we extend $m=0$ in Equation (32a, b), then two-dimensional solution for steady flow and continuous input concentration of uniform nature may be given as:

$$C(Z,t)=\frac{\mu_0}{\gamma_1}+\left(C_0-\frac{\mu_0}{\gamma_1}\right)F(Z,t)+\left(C_i-\frac{\mu_0}{\gamma_1}\right)G(Z,t) \tag{39}$$

where

$$F(Z,t)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_1D_0\right)^{1/2}\right\}Z}{2\ D_0}\right]erfc\left[\frac{R_0\ Z-\left(U_0^2+4\gamma_1D_0\right)^{1/2}\ t}{2\sqrt{D_0\ R_0T}}\right]$$
$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_1D_0\right)^{1/2}\right\}Z}{2\ D_0}\right]erfc\left[\frac{R_0\ Z+\left(U_0^2+4\gamma_1D_0\right)^{1/2}\ t}{2\sqrt{D_0\ R_0T}}\right]$$

$$G(Z,T)=\exp\left(-\frac{\gamma_1}{R_0}t\right)\left\{1-\frac{1}{2}erfc\left(\frac{R_0Z-U_0t}{2\sqrt{D_0R_0t}}\right)-\frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{R_0Z+U_0t}{2\sqrt{D_0R_0t}}\right)\right\}$$

$$Z=\frac{\log(1+ax)}{a}+\frac{\log(1+by)}{b},\ D_0=D_{x_0}+D_{y_0},U_0=u_0+v_0-n(aD_{x_0}+bD_{y_0}),$$
$$\gamma_1=\gamma_0+n(au_0+bv_0).$$

If we put $a=0,b=0$ and $m=0$, in the Equation (32a,b) it shows good agreement with result obtained by Al-Niami & Rushton (1977) for constant coefficients and may be written as:

$$C(Z,t)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F(Z,t)+\left(C_i-\frac{\mu_0}{\gamma_0}\right)G(Z,t)\qquad;\quad 0<t\le t_0 \tag{40.a}$$

$$C(Z,t)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F(Z,t)-C_0\,F(Z,t-t_0)+\left(C_i-\frac{\mu_0}{\gamma_0}\right)G(Z,t)\qquad;\quad t>t_0 \tag{40.b}$$

where

$$F(Z,t)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_0 D_0\right)^{1/2}\right\}Z}{2\,D_0}\right]erfc\left[\frac{R_0 Z-\left(U_0^2+4\gamma_0 D_0\right)^{1/2}t}{2\sqrt{D_0 R_0 t}}\right]$$
$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_0 D_0\right)^{1/2}\right\}Z}{2\,D_0}\right]erfc\left[\frac{R_0 Z+\left(U_0^2+4\gamma_0 D_0\right)^{1/2}t}{2\sqrt{D_0 R_0 t}}\right]$$

$$G(Z,t)=\exp\left(-\frac{\gamma_0}{R_0}t\right)\left\{1-\frac{1}{2}erfc\left(\frac{R_0 Z-U_0 t}{2\sqrt{D_0R_0t}}\right)-\frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{R_0 Z+U_0 t}{2\sqrt{D_0R_0t}}\right)\right\}$$

$Z=x+y$, $D_0=D_{x_0}+D_{y_0}$, $U_0=u_0+v_0$

If we put $y=0$ in Equation (32a,b) i.e. all terms corresponding to y axis are taken to be zero, the solution may be written in one-dimension for pulse type input point source as:

$$C(X,T)=\frac{\mu_0}{\gamma_1}+\left(C_0-\frac{\mu_0}{\gamma_1}\right)F(X,T)+\left(C_i-\frac{\mu_0}{\gamma_1}\right)G(X,T)\quad;\quad 0<T\le T_0 \tag{41.a}$$

$$C(X,T)=\frac{\mu_0}{\gamma_1}+\left(C_0-\frac{\mu_0}{\gamma_1}\right)F(X,T)-C_0\,F(X,T-T_0)+\left(C_i-\frac{\mu_0}{\gamma_1}\right)G(X,T)\quad;\quad T>T_0 \tag{41.b}$$

where

$$F(X,T)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}X}{2\,D_0}\right]erfc\left[\frac{R_0\,X-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}T}{2\sqrt{D_0 R_0T}}\right]$$
$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}X}{2\,D_0}\right]erfc\left[\frac{R_0\,X+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}T}{2\sqrt{D_0 R_0T}}\right]$$

$$G(X,T)=\exp\left(-\frac{\gamma_1}{R_0}T\right)\left\{1-\frac{1}{2}erfc\left(\frac{R_0X-U_0T}{2\sqrt{D_0R_0T}}\right)-\frac{1}{2}\exp\left(\frac{U_0}{D_0}X\right)erfc\left(\frac{R_0X+U_0T}{2\sqrt{D_0R_0T}}\right)\right\}$$

$$X=\frac{\log(1+a\,x)}{a},\quad T=\int_0^t f(mt)\,dt,\; D_0=D_{x_0},\; U_0=u_0-n\,a\,D_{x_0},\quad \gamma_1=\gamma_0+na\,u_0.$$

RESULTS AND DISCUSSIONS

The concentration values obtained from the solution Equation (32.a) in the presence of the source, in the time domain $t\le t_0$ is discussed graphically for a chosen set of data taken from the published experimental and theoretical literatures. The concentration values C/C_0 are evaluated assuming reference concentration as $C_0=1.0$, in a finite domain along longitudinal and transverse directions $0\le x(m)\le 5$ and $0\le y(m)\le 3$ respectively. The medium is supposed to be heterogeneous along both the directions. The source of the input pollutant is assumed to be eliminated at $t_0=6$ day, beyond this time the concentrations values evaluated from the Equation (32.b). The obtained solution is demonstrated graphically with two forms of ground water velocity namely sinusoidal time varying velocity $f(mt)=1-\sin(mt)$ and exponential decreasing form $f(mt)=\exp(-mt)$. In ground water, water level may demonstrate seasonally sinusoidal behaviour (Kumar & Kumar, 1998). In real scenario, the solute mass dissipates in both directions longitudinal as well as lateral direction but the dissipation along the lateral direction may be much less than in comparison to the longitudinal direction. Considering this fact lateral velocity is considered one-tenth of longitudinal. The common input parameters are taken $C_0=1$, $C_i=0.1$, $a=0.01(m^{-1})$, $b=0.01(m^{-1})$, $u_0=1.05(m/day)$, $v_0=0.105(m/day)$, $\gamma_0=0.04(day^{-1})$ and $\mu_0=0.0021(kg/m^3day)$. The common input parameters for Figure 1a, 1b, 2a and 2b are $D_{x_0}=1.25(m^2/day)$, $D_{y_0}=0.125(m^2/day)$ and $m=0.1(day^{-1})$.

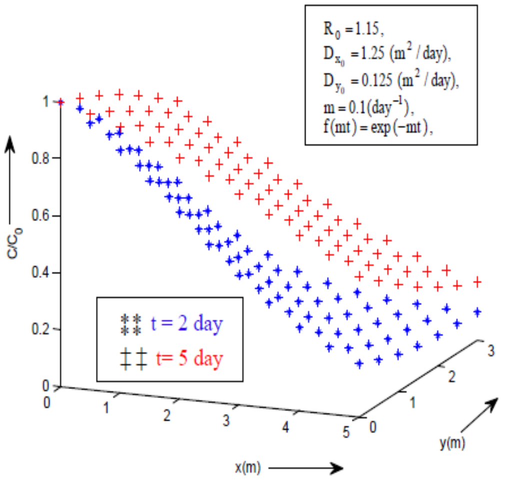

Fig. 1(a): Comparison of solution Eq.(32.a), for different time for exponentially decreasing velocity $f(mt)=\exp(-mt)$.

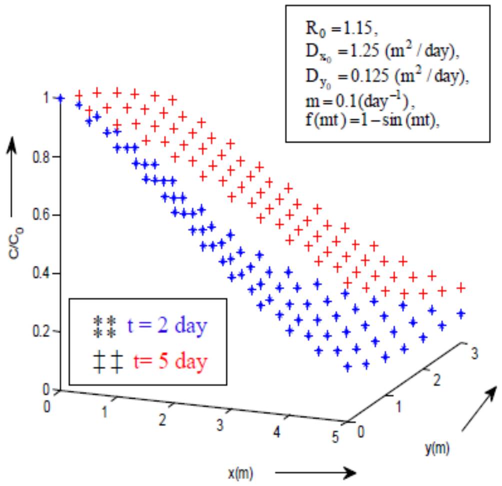

Fig. 1(b): Comparison of solution Eq.(32.a), for different time for sinusoidal velocity $f(mt)=1-\sin(mt)$.

Figure 1(a) and 1(b) are drawn for exponentially decreasing and sinusoidal form of velocity respectively at different time $t(days)=2$ and 5. In both figures it reveals that concentration profiles at particular position for exponentially decreasing form of groundwater velocity are lower for smaller time and higher for larger time, but in comparison to sinusoidal form velocity the concentration levels are higher for all time at the same position. It also illustrates that in the exponential form of velocity, the rehabilitation process is faster than sinusoidal form. In both form of velocity the concentration at origin $x=0$ and $y=0$ are equals to 1.

Figure 2(a) and 2(b) illustrate the effect of various retardation factor $R_0=1.15$ and 1.85 on the concentration profile at time $t=5\ (day)$ and ground water velocities are taken same as in Figure 1a and 1b, respectively. In both form of velocity the concentration profile are lower for higher and higher for lower retardation factor. It means that pollutant rehabilitate slowly for higher retardation value.

The common input parameters values considered for Figure 3a, 3b, 4a and 4b are $t=5\ (day)$ and $R_0=1.15$

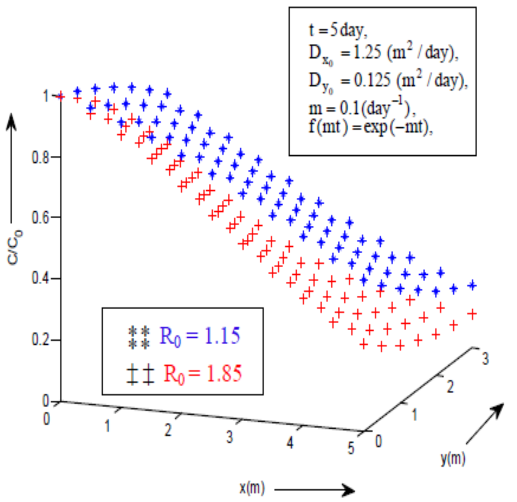

Fig. 2(a): Comparison of solution Eq.(32.a), for different retardation factor for exponentially decreasing velocity $f(mt) = \exp(-mt)$.

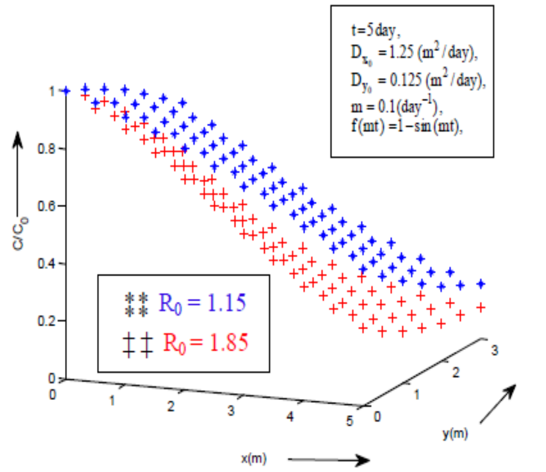

Fig. 2(b): Comparison of solution Eq. (32.a), for different retardation factor for sinusoidal form of velocity $f(mt) = 1 - \sin(mt)$.

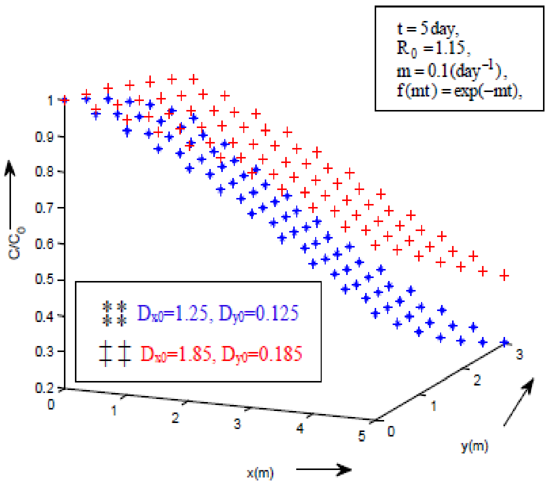

Fig. 3(a): Effect of different dispersion coefficient on solute transport described by solution Eq.(32.a). where $f(mt) = \exp(-mt)$.

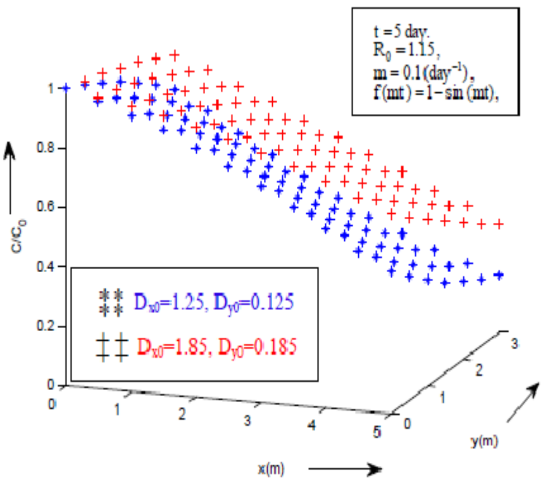

Fig. 3(b): Effect of different dispersion coefficient on solute transport described by solution Eq.(32.a). where $f(mt) = 1 - \sin(mt)$.

Figure 3(a) and 3(b) demonstrates effect of different dispersion coefficient $D_{x_0} = 1.25(m^2/day)$, $D_{y_0} = 0.125(m^2/day)$ and $D_{x_0} = 1.85(m^2/day)$, $D_{y_0} = 0.185(m^2/day)$ on concentration profiles at unsteady parameter $m = 0.1(day^{-1})$ and ground water velocity are taken same form as in Figure 1a and 1b, respectively. It reveals that for both form of seepage velocity the concentration levels are lower for lower and higher for higher dispersion coefficient.

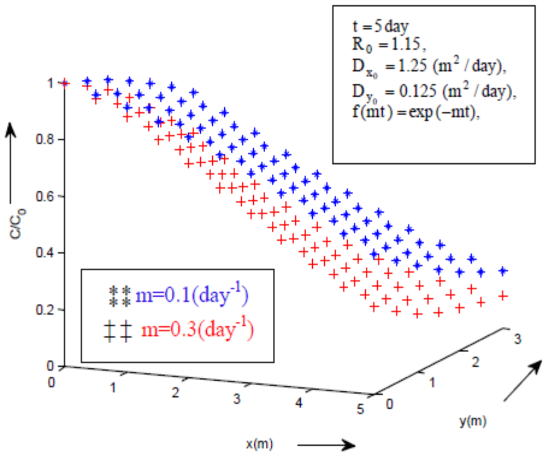

Figure 4(a): Effect of for different unsteady parameter on solute transport described by solution Eq.(32.a). where $f(mt) = \exp(-mt)$.

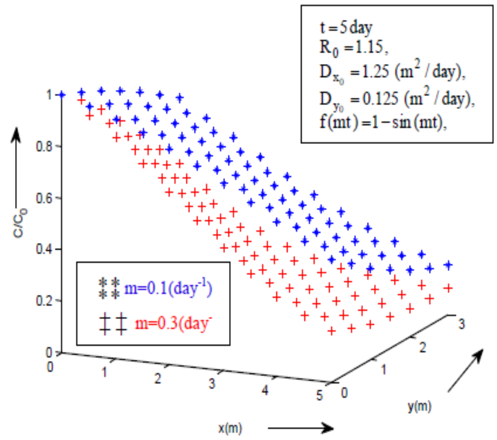

Fig. 4(b): Effect of for different unsteady parameter on solute transport described by solution Eq.(32.a). where $f(mt) = 1 - \sin(mt)$.

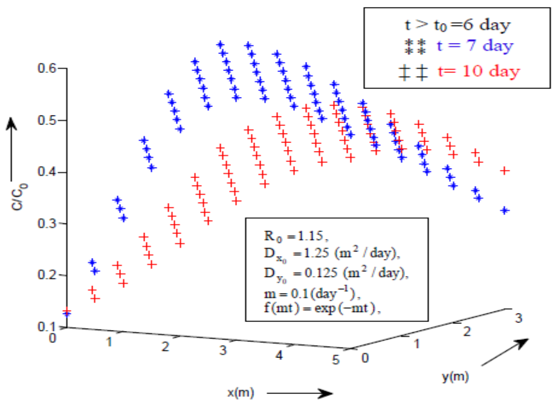

Fig. 5(a): Comparison of solute concentration for different time due to $f(mt) = \exp(-mt)$ described by solution Eq.(32.b).

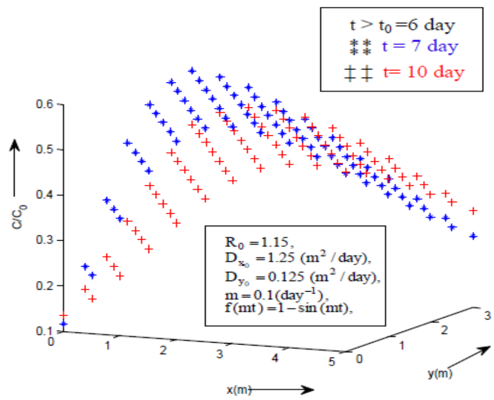

Fig. 5(b): Comparison of solute concentration for different time due to $f(mt) = 1 - \sin(mt)$ described by solution Eq.(32.b).

Figure 4(a) and 4(b) illustrate the effect of various unsteady parameter $m(day^{-1}) = 0.1$ and 0.3 on the concentration profiles at particular dispersion coefficient $D_{x_0} = 1.25(m^2/day)$, $D_{y_0} = 0.125(m^2/day)$. It demonstrates that concentration level at particular position is lower for higher unsteady parameter and higher for lower unsteady parameter. This phenomenon remains same for both form of velocity but sinusoidal form of velocity the rehabilitation rate are faster than exponential form.

Figure 5(a), 5(b), 6(a), 6(b), 7(a), 7(b) and 8(a), 8(b) are drawn for the solution in Eq.(32.b) when pollutants are not entering in the domain. Figure 5(a) and 5(b) are drawn at different time $t(day) = 7$ and 10 and rest parameters and ground water velocity are taken same form same as Figure 1(a) and 1(b), respectively. It reveals that near the source boundary the concentration levels initially increases for both form of groundwater velocity up to certain distance and then it deceases slowly with space.

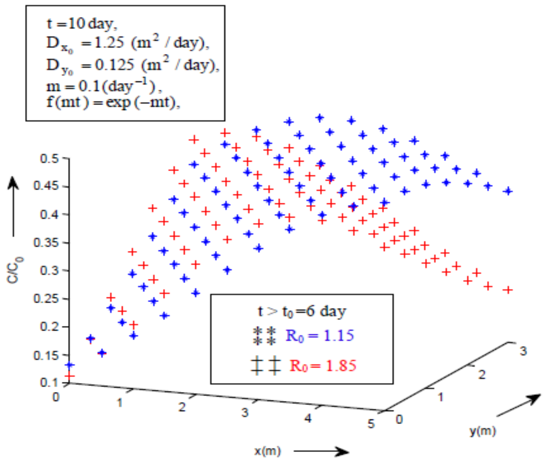

Fig. 6(a): Comparison of solute concentration for different retardation coefficient due to $f(mt) = \exp(-mt)$ described by solution Eq.(32.b).

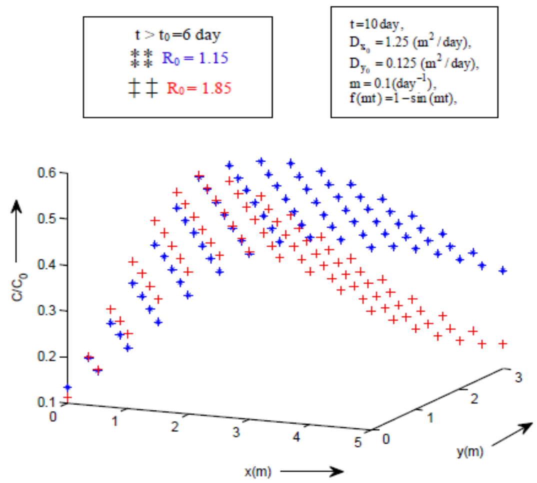

Fig. 6(b): Comparison of solute concentration for different retardation coefficient due to $f(mt) = 1 - \sin(mt)$ described by solution Eq.(32.b).

Figure 6(a) and 6(b) are drawn at different retardation factors at time $t = 10\ (day)$ and rest parameters and ground water velocity are taken same form as Figure 2(a) and 2(b), respectively. It reveals that near source boundary the concentration levels initially increases for both form and then deceases slowly with space, but for higher retardation factor the pollutants rehabilitates faster. At particular position the concentration level near the boundary is lower for higher retardation factor and higher for lower retardation factor in both forms of velocity but after certain distance travelled from boundary the concentration level is also lower for higher retardation factor and higher for lower retardation factor.

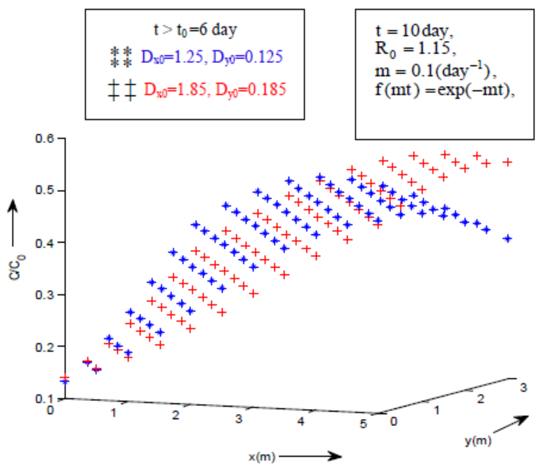

Fig. 7(a): Comparison of solute concentration for different dispersion coefficient due to $f(mt) = \exp(-mt)$ described by solution Eq.(32.b).

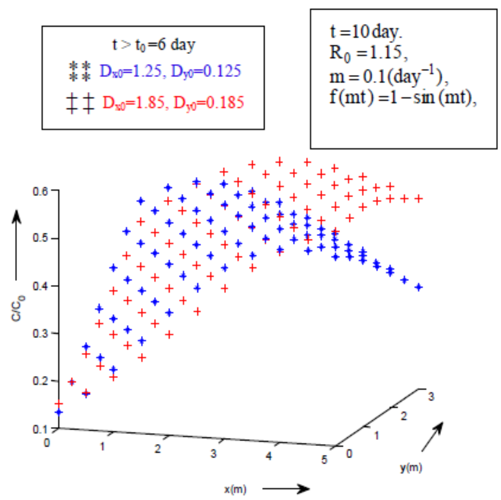

Fig. 7(b): Comparison of solute concentration for different dispersion coefficient due to $f(mt) = 1 - \sin(mt)$ described by solution Eq.(32.b).

Figure 7(a) and 7(b) are drawn at various dispersion coefficient $D_{x_0} = 1.25(m^2/day)$, $D_{y_0} = 0.125(m^2/day)$ and $D_{x_0} = 1.85(m^2/day)$, $D_{y_0} = 0.185(m^2/day)$ on concentration profiles at particular time $t = 10\ (day)$ and rest parameters and ground water velocity are taken same form as Figure 3a and 3b, respectively. It reveals that near the source boundary the concentration levels initially increases for both functions and after some distance travelled it deceases but decreasing levels of concentration are lower for lower dispersion coefficient.

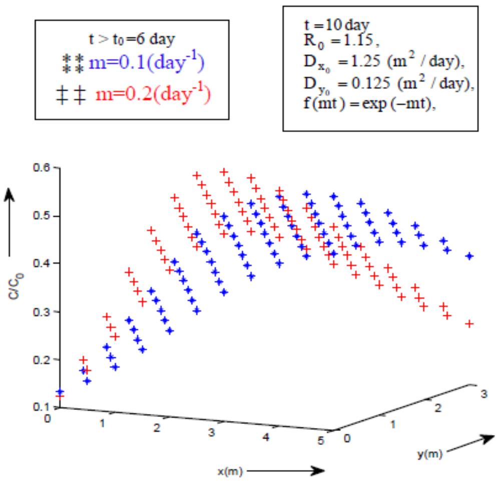

Fig. 8(a): Comparison of solute concentration for different unsteady parameter due to $f(mt) = \exp(-mt)$ described by solution Eq.(32.b).

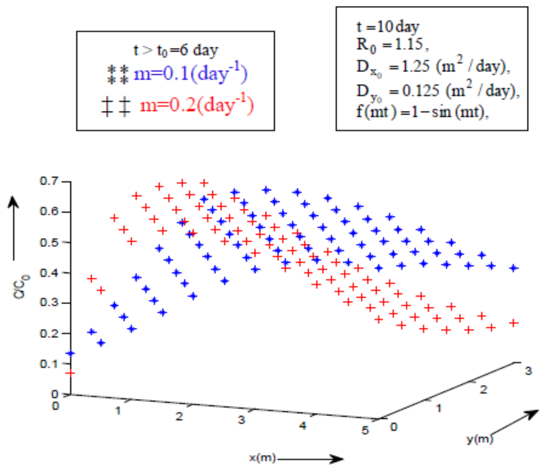

Fig. 8(b): Comparison of solute concentration for different unsteady parameter due to $f(mt) = 1 - \sin(mt)$ described by solution Eq.(32.b).

Figure 8(a) and 8(b) are drawn at various unsteady parameter $m(day^{-1}) = 0.1$ and 0.2 on concentration profiles at particular time $t = 10\ (day)$ and rest parameters and ground water velocity are taken same form are same as Figure 4a and 4b, respectively. It reveals that near the source boundary the concentration levels initially increases up to certain distance for both function then deceases but decrease level of concentration are lower for higher unsteady parameter. This phenomenon remains same for both form of groundwater velocity but sinusoidal form of velocity the rehabilitation rate are faster than exponential form. It is ascertained that the contaminant concentration decreases in both longitudinal and lateral directions with time and distance travelled in presence of source contaminant. While in the absence of source contaminants, it increases and goes on increasing which arrive towards maximum and then starts decreases and goes on decreasing which arrive towards minimum or harmless concentration. This decreasing inclination of contaminant concentration with time and distance travelled may help to rehabilitate the contaminated ground water table.

The derived mathematical model can be extended in three dimensions which may be given by Equations (42.a, b) as

$$C(Z,T) = \frac{\mu_0}{\gamma_1} + \left(C_0 - \frac{\mu_0}{\gamma_1}\right)F(Z,T) + \left(C_i - \frac{\mu_0}{\gamma_1}\right)G(Z,T) \quad ; \quad 0 < T \le T_0 \qquad (42.a)$$

$$C(Z,T)=\frac{\mu_0}{\gamma_1}+\left(C_0-\frac{\mu_0}{\gamma_1}\right)F(Z,T)-C_0F(Z,T-T_0)+\left(C_i-\frac{\mu_0}{\gamma_1}\right)G(Z,T) \qquad ; \quad T>T_0 \tag{42.b}$$

where

$$F(Z,T)=\frac{1}{2}\exp\left[\frac{\left\{U_0-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}Z}{2\,D_0}\right]erfc\left[\frac{R_0\,Z-\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\,T}{2\sqrt{D_0\,R_0T}}\right]$$

$$+\frac{1}{2}\exp\left[\frac{\left\{U_0+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\right\}Z}{2\,D_0}\right]erfc\left[\frac{R_0\,Z+\left(U_0^2+4\gamma_1 D_0\right)^{1/2}\,T}{2\sqrt{D_0\,R_0T}}\right]$$

$$G(Z,T)=\exp\left(-\frac{\gamma_1}{R_0}T\right)\left\{1-\frac{1}{2}erfc\left(\frac{R_0Z-U_0T}{2\sqrt{D_0R_0T}}\right)-\frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{R_0Z+U_0T}{2\sqrt{D_0R_0T}}\right)\right\}$$

$$Z=\frac{\log(1+a\,x)}{a}+\frac{\log(1+b\,y)}{b}+\frac{\log(1+c\,z)}{c},$$

$$T=\int_0^t f(mt)\,dt\,,\ D_0=D_{x_0}+D_{y_0}+D_{z_0}\,,$$

$$U_0=u_0+v_0+w_0-n(a\,D_{x_0}+b\,D_{y_0}+c\,D_{z_0})\ ,\ \gamma_1=\gamma_0+n(a\,u_0+b\,v_0+c\,w_0).$$

where new notations D_{z_0}, w_0 and c represents initial dispersion coefficient, unsteady uniform seepage velocity and heterogeneity parameter respectively, along the direction perpendicular to both longitudinal and transverse directions or water table.

CONCLUSIONS

This study mainly concerns the development of a new analytical solution of the advection-dispersion equation in two-dimensions by taking into account a semi-infinite porous domain and a point-like injection, with a variable dispersion coefficient for non-reactive contaminant transport. The pulse type boundary conditions are considered in the aquifer system. Due to the effects of the boundary condition and flow velocity, the amount of solute retained decreases with time and position. The solutions are obtained for sinusoidal and exponential decreasing form of velocity which represents the seasonal pattern in tropical regions. Analytical solution for this hypothetical scenario, which is based on the assumption that the contaminant is distributed exponentially decreasing function of position throughout the domain, can be used as a benchmark tool for analytical analyses and may helpful to predict the concentration levels at space and time which may help to reduce/eliminate the concentration. Some known solutions are derived as particular cases from the solutions of the present paper. It accomplishes the validation of the mathematical formulations and analytical procedures obtaining the solutions. The obtained solutions show a good applicability to real cases of solute transport phenomenon. An analytical solution is very important and economical because it provide better physical insight into the water and solute transport phenomenon and it is also needed as validation test for numerical schemes. The analytical solutions developed in the present study are apropos to more general hydrological conditions influencing the solute transport in groundwater originating from pulse type input point sources.

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REFERENCES

Al-Niami, A. N. S. and Rushton, K. R. (1977). Analysis of flow against dispersion in porous media. J. Hydrol., 33; 87-97.

Aral, M. M. and Liao, B. (1996). Analytical solutions for two-dimensional transport equation with time-dependent dispersion coefficients. J. Hydrol. Eng, 1(1); 20-32.

Banks, R. B. and Ali, J. (1964). Dispersion and adsorption in porous media flow. J. Hydrol. Div., 90; 13-31.

Bing, B., Huawei, L., Tao, X. and Xingxin, C. (2015). Analytical solutions for contaminant transport in a semi-infinite porous medium using the source function method. Comp. and Geotech., 69; 114-123.

Carnahan, C. L. and Remer, J. S. (1984). Non-equilibrium and equilibrium sorption with a linear sorption isotherm during mass transport through porous medium: Some analytical solutions. J. Hydrology (Amsterdam, Neth), 73; 227-258.

Chen, J. S. and Liu, C. W. (2011). Generalized analytical solution for advection-dispersion equation in finite spatial domain with arbitrary time-dependent inlet boundary condition. Hydrol. Earth Sys. Sci., 15; 2471-2479.

Chen, J. S., Liu, C. W., Hsu, H. T. and Liao, C. M. (2003). A Laplace transformed power series solution for solute transport in convergent flow field with scale-dependent dispersion. Water Resou. Res., 39(8); 14-1–14-10.

Chen, J. S., Ni, C. F. and Liang, C. P. (2008). Analytical power series solution to the two-dimensional advection-dispersion equation with distance-dependent dispersivity. Hydrol. Process., 22(24); 4670-4678.

Crank, J. (1975). The Mathematics of Diffusion. Oxford Univ. Press London, 2nd ed.

Das, P., Akhter, A. And Singh, M. K. (2018). Solute transport modelling with the variable temporally dependent Boundary. Sådhanå; 43:12

Djordjevich, A. and Savovic, S. (2017). Finite difference solution of two-dimensional solute transport with periodic flow in homogeneous porous media. J. Hydrol. and Hydromech., 65(4); 426-432.

Hunt, B. (1998). Contaminant source solutions with scale-dependent dispersivities. J. Hydrol. Eng., 3(4); 268-275.

Jaiswal, D. K., Kumar, A., Kumar, N. and Singh, M. K. (2011). Solute Transport along Temporally and Spatially Dependent Flows through Horizontal Semi-Infinite Media: Dispersion Proportional to Square of Velocity. J. Hydrol. Eng, 16(3); 228-238.

Kumar, A. and Yadav, R. R. (2015). One-dimensional solute transport for uniform and varying pulse type input point source through heterogeneous medium. Environ. Tech., 36(4); 487-495.

Kumar, A., Jaiswal, D. K. and Kumar, N. (2010). Analytical solutions to one-dimensional advection–diffusion equation with variable coefficients in semi-infinite media. J. Hydrol., 380; 330-337.

Kumar, N. and Kumar, M. (1998). Solute dispersion along unsteady groundwater flow in a semi-infinite aquifer. Hydro. and Earth System Sciences, 2; 93-100.

Majdalani, S., Chazarin, J. P., Delenne, C. and Guinot, V. (2015). Solute transport in periodical heterogeneous porous media: Importance of observation scale and experimental sampling. J. Hydrol., 520; 52-60.

Marino, M. A. (1974). Distribution of contaminants in porous media flow. Water Resour. Res., 10; 1013-1018.

Ogata, A. (1970). Theory of dispersion in granular media. US Geological Survey Professional Papers, 411-I: 34.

Sanskrityayn, A., Bharati , V. K. and Kumar, N. (2018). Solute transport due to spatio-temporally dependent dispersion coefficient and velocity: analytical solutions. J. Hydrol. Eng., 23(4); 04018009.

Sanskrityayn, A., Bharati, V. K. and Kumar, N. (2016). Analytical solution of ADE with spatiotemporal dependence of dispersion coefficient and velocity using Green's function method. J. Groundwater Res., 5(1); 24-31.

Singh, M. K. Kumari, P. and Mahato, N. K. (2013). Two-dimensional solute transport in finite homogeneous porous formations. Int. J. Geology, Earth & Environ. Sciences, 3(2); 35-48.

Smedt, F. D. (2006). Analytical solutions for transport of decaying solutes in rivers with transient storage. J. Hydrol., 330(3-4); 672-680.

Su, N., Sander, G. C., Liu, F., Anh, V. and Barry, D. A. (2005). Similarity solutions for solute transport in fractal porous media using a time and scale-dependent dispersivity. App. Mathematical Model., 29; 852-870.

Sudicky, E. A. (1986). A natural gradient experiment on solute transport in a sand aquifer: Spatial variability of hydraulic conductivity and its role in the dispersion process. Water Resour. Res., 22(13); 2069-2082.

Sykes, J. F., Pahwa, S. B., Lantz, R. B. and Ward, D. S. (1982). Numerical simulation of flow and contaminant migration at an extensively monitored landfill. Water Resour. Res., 18(6); 1687-1704.

Van-Genuchten, M. T. and Alves, W. J. (1982). Analytical solutions of the one-dimensional convective-dispersive solute transport equation. U S Dept. Ag. Tech. Bull. No. 1661; 1-51.

Yadav, R. R. and Kumar, L. K. (2018). Two-dimensional conservative solute transport with temporal and scale-dependent dispersion: Analytical solution. Int. J. Adv. in Math., 2018(2); 90-111.

Yadav, R. R. and Jaiswal, D. K. (2011). Two-dimensional analytical solutions for point source contaminants transport in semi- infinite homogeneous porous medium. J. Eng. Sci. and Tech., 6(4); 459-468.

Yadav, R. R., Jaiswal, D. K., Yadav, H. K. and Gulrana. (2011). Temporally dependent dispersion through semi-infinite homogeneous porous media: an analytical solution. IJRRAS, 6 (2); 158-164.

Yadav, S. K., Kumar, A. and Kumar, N. (2012). Horizontal solute transport from a pulse type source along temporally and spatially dependent flow: Analytical solution. J. Hydro., 412-413; 193-199.

Yates, S. R. (1990). An analytical solution for one-dimension transport in heterogeneous porous media. Water Resour. Res., 26(10); 2331-2338.

Yates, S. R. (1992). An analytical solution for one-dimension transport in porous media with an exponential dispersion function. Water Resour. Res., 28(8); 2149-2154.

Zhan, H., Wen, Z., Huang, G. and Sun, D. (2009). Analytical solution of two dimensional solute transports in an aquifer-aquitard system. J. Contaminant Hydrol., 107; 162-174.