



Analytical Solutions for Solute Transport from two-point Sources along Porous Media Flow with Spatial Dispersivity involving Flexible Boundary Inputs, initial Distributions and Zero-order Productions

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Abstract

This study derives an analytical solution of a one-dimensional (1-D) Advection-Dispersion Equation (ADE) for solute transport with two contaminant sources incorporating the source term. Groundwater velocity is considered as a linear function of space while the dispersion as a n^{th} power of velocity and analytical solutions are obtained for $n = 1.0, 1.5$ and 2.0 . The solution is derived using the Generalized Integral Transform Technique (GITT) with a new regular Sturm-Liouville Problem (SLP). Analytical solutions are compared with numerical solutions obtained in MATLAB *pedpe* solver and are found to be in good agreement. The obtained solutions are illustrated for linear combination of exponential input distribution and its particular cases. The dispersion coefficient and temporal variation of the source term on the solute distribution are demonstrated graphically for the set of input data based on similar data available in the literature. As an illustration, model predictions are used to estimate the time histories of the radiological doses of uranium at different distances from the sources boundary in order to understand the potential radiological impact on the general public for such problem.

Keywords: Advection, Dispersion, Groundwater, Analytical solution, Contaminant, Two sources.

INTRODUCTION

The pollution of groundwater has seriously increased over the years as a result of socioeconomic development, and interests many Scientists. The groundwater can be contaminated through the infiltration leachate process causing a reduction of groundwater flow rate, which results in difficult rehabilitation of groundwater. This pollution has a serious impact on human lives, and activities as well as productivity. Thus, it is very important to study the solute transport characteristics in porous media for human and ecological health. The transport of a solute in porous media is governed by the processes of advection and the dispersion. The mediums through which the solute transport occurs are homogeneous or heterogeneous. This characteristic of media as an important role to the solute transport through them (Kumar & Yadav, 2014; Sanskrityayn et al., 2018; Yadav & Kumar, 2019). The advection-dispersion equation can be solved numerically or analytically. Analytical solutions are still pursued by many scientists because they are relatively transparent with respect to model inputs and outputs, and they can provide better physical insights into the problems (Park & Zhan, 2001).

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Many analytical solutions of contaminants transport in subsurface of water considering adsorption are present in the literature. Using the Laplace Transform Technique (LTT), van Genuchten (1981) obtained analytical solutions for ADE with zero-order production, simultaneous adsorption, and first-order decay for chemical transport. van Genuchten & Alves (1982) proposed several solutions that quantitatively describe the behaviour of solute in surface/subsurface in finite and semi-infinite medium considering adsorption, first-order decay and zero-order production in homogeneous medium. Considering the dispersion coefficient as a linear and exponential increasing space function, Yates (1992) developed an analytical solution of advection dispersion equation in one-dimension. Zoppou & Knight (1997) obtained an analytical solution for the space dependent dispersion. Su et al. (2005) developed an analytical solution to ADE with space-time dependent dispersion coefficient for predicting solute transport in a steady and saturated subsurface flow through heterogeneous porous media. Zhan et al. (2009) obtained an analytical solution for 2-D solute transport using first and third type boundary conditions. Mazarheti et al. (2013) derived analytical solution of one-dimensional solute transport with several point sources and arbitrary time-dependent emission rate. Its analytical solution was valid only for constant-parameters. Using the LTT, Kumar & Yadav (2014) developed a 1-D analytical solution for conservative solute transport in heterogeneous porous media for pulse type input point source. Kumar et al. (2019) studied the effect of the source/sink term on the solute.

One of the methods commonly used to solve the ADE in finite domain with distance dependent coefficients is the GITT. The GITT was applied by Liu et al. (2000) to solve the one-dimensional ADE in heterogeneous porous media with source/sink term, coupled with either linear or nonlinear sorption and decay. The GITT coupled with the LTT were used by Chen & Liu, (2011) to solve a 1-D ADE in a finite spatial domain with an arbitrary time-dependent inlet boundary condition. For a finite spatial domain, the 1-D ADE considering the sorption and desorption of solute, with arbitrary space dependent coefficients was solved analytically using the GITT (Skaggs et al., 2007). Pérez Guerrero et al., (2009) presented a new analytical method to solve a 3-D ADE in a finite domain with time varying boundary condition for both transient and steady-state regimes using change of variables in combination with the Classic Integral Transform Technique (CITT). Chen et al. (2011) presented an analytical solution of two-dimensional ADE in cylindrical coordinates using a combination of the second kind finite transform method and the GITT. Recently, Bharati et al. (2017); Bharati et al. (2018) and Bharati et al. (2019) presented an analytical solution of solute transport with distance depending coefficients without source term, using the GITT with a new regular SLP with a self-adjoint operator to derive analytical solutions in a finite domain. Although these studies proposed novel methods to solve the ADE, they did not incorporate several parameters on the solute transport such as the source term and the presence of several pollutant sources. Including many parameters in pollutant transport equation is helpful to cover many aspects of contaminant transport in groundwater in more natural way (Chaudhary et al., 2020).

The aim of this study is to investigate analytically for the 1-D ADE in heterogeneous domain, with space and time dependent production term and two inputs localized at the boundaries of the domain. The analytical solution is obtained using the GITT with associated advection-dispersion SLP using a self-adjoint operator. The analytical solution is validated with the help of the numerical solution. To our knowledge, no such results have established previously for this type of problem. The main focus of this study was to investigate the effect of the two points input sources, the additional source term and the degree of heterogeneity on contaminant distribution. The experimental results for this type of problem are not available in the literature for verification. However, this study is likely to present a real scenario of groundwater contamination. The developed analytical model is applied to illustrate the time variation of uranium radioactivity concentration and the potential radiological impact through the time histories of the radiological doses at different distances from the origin.

MATERIALS AND METHODS

In this study, we consider one-dimensional solute transport in a horizontal aquifer. The one-dimensional ADE in general form, describing a non-conservative solute transport in a finite porous media domain with spatially dependent dispersion coefficient, $D(x)[L^2T^{-1}]$, Darcy velocity $u(x)[LT^{-1}]$, and zero-order production may be written as Chaudhary et al. (2020):

$$\theta \frac{\partial C}{\partial t} + f \frac{\rho_b}{\eta} \frac{\partial S}{\partial t} = \frac{\partial}{\partial x} \left[D(x) \frac{\partial C}{\partial x} - u(x)C \right] - \mu_l C - \mu_s \rho_b S + \gamma(x, t) \quad (1)$$

where $C(x, t)$ is the solute concentration in the liquid phase $[ML^{-3}]$ at a position $x[L]$ of the medium at a time $t[T]$, $S(x, t)$ is the solid phase concentration $[ML^{-3}]$, representing the sorbed solute per unit mass of solid, θ is the volumetric water content, f represents the fraction of sorption site, ρ_b is the bulk density of porous media $[ML^{-3}]$, η is the porosity of the porous media, μ_l and μ_s are the first order decay rate of the liquid and solid phases concentration respectively, $\gamma(x, t)$ stands for an arbitrary space and time variable zero-order production. The left-hand side of Eq. (1) represents change in solute concentration in liquid and solid phases with time respectively for the first and second terms. The right-hand side of the Eq. (1) represents the influence of the dispersion on the solute concentration distribution by the first term and the change of the solute concentration due to advective solute transport by the second term. The third and the fourth terms of the right-hand represent the first-order decay of solute in the liquid and solid phases in the medium respectively. The fifth term represents the zero-order production $\gamma(x, t) > 0$ or sink $\gamma(x, t) < 0$ for solute which represents internal/external production or sink of the solute in the medium.

Furthermore, we assumed that the expression for a linear equilibrium between the solute substances in the solid-liquid phase is given by Sim & Chrysikopoulos (1996) and Singh & Das (2015):

$$S = Fk_d C \quad (2)$$

where F represents the mass fraction of sorption particles where sorption is instantaneous, K_d refers to the distribution coefficient $[L^3M^{-1}]$. For an instantaneous sorption in all the mass $F = 1$.

In this study, the groundwater velocity is assumed to be governed by the Darcy equation. Due to the steady recharge, groundwater velocity will be increasing linearly with position (Serrano, 1992). Hence both the dispersion coefficient and velocity are considered spatially dependant in general form. In this study, the expression of groundwater velocity and dispersion coefficient are considered to follow the dispersion theory according to which the dispersion is proportional to the n^{th} -power of the space velocity ($D \propto u^n$) (Freeze & Cherry, 1979), where n is considered as 1.0, 1.5, and 2.0. Thus, the expression of velocity and hydrodynamic dispersion coefficients are considered as follows:

$$u(x, t) = u_0(a_1 + a_2x) \text{ and } D(x, t) = (D_0 + D^* \tau)(a_1 + a_2x)^n \quad (3)$$

where τ is the tortuosity, $D_0[L^2T^{-1}]$, $D^*[L^2T^{-1}]$ and $u_0[LT^{-1}]$ are respectively constant mechanical dispersion coefficient, molecular diffusion coefficient and velocity in a steady flow domain through a homogeneous porous medium. a_1 is a non-dimensional parameter and $a_2[L^{-1}]$ represents the heterogeneity parameter.

In this study, the source/sink is expressed by single function on space and time-dependent as

that proposed by Kumar et al. (2019):

$$\gamma(x,t) = \gamma_0 p(x)q(t) \tag{4}$$

where γ_0 is the uniform zero-first order production coefficient [$ML^{-3}T^{-1}$].
By substituting Eqs. (2), (3), and (4) into Eq. (1) we get:

$$R \frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left[D_{x_0} (a_1 + a_2 x)^n \frac{\partial C}{\partial x} - u_0 (a_1 + a_2 x) \right] - \mu C + q_0 p(x)q(t) \tag{5}$$

where $R = \theta + f(\rho_b/\eta) Fk_d$ is the retardation factor and $\mu = \mu_l + \mu_s \rho_b Fk_d$.

Before pollutant sources are injected in the medium, it is supposed to contain a background contamination. This contamination can be expressed as a linear combination of some input contaminant concentration in liquid and solid phases respectively (Chaudhary et al., 2020). In this study, it is assumed that previous concentration in the liquid phase is an arbitrary space variable function, while the input concentration in the solid phase is uniform with distance. The initial condition may be written in general form as follows.

$$C(x,t) = h(x) + K_s^i; \quad x, \geq 0, \quad t = 0 \tag{6}$$

where $h(x)$ is the input liquid phase concentration and K_s^i the input solid phase concentration.

The geological formation of the medium through which the dispersion occurs is considered to be bounded by two parallel planes, e.g. the planes at $x = 0$, $x = l$ with two pollutant sources located at the two planes so that the pollutant enters through the planes (see Fig. (1)).

As the groundwater flows from $x = 0$ to $x = l$, this situation could be referred to advection-dispersion with simultaneous input in the flow direction and against the direction of the flow. In this study, the surface concentrations variables are considered and may be expressed in general form as:

$$C(x,t) = f_1(t), \quad x = 0, \quad t > 0 \tag{7}$$

$$C(x,t) = f_2(t), \quad x = l, \quad t > 0 \tag{8}$$

where $f_1(t)$ and $f_2(t)$ are the amount of pollutant concentration entering through the planes $x = 0$ and $x = l$ respectively. The initial and the boundary conditions Eqs. (6)-(8) are similar to the situation of the diffusion into a plane sheet of material with surfaces, $x = 0$, $x = l$, maintained at constant concentrations C_1 and C_2 respectively, having a general initial heat distribution studied by Cranck (1975).

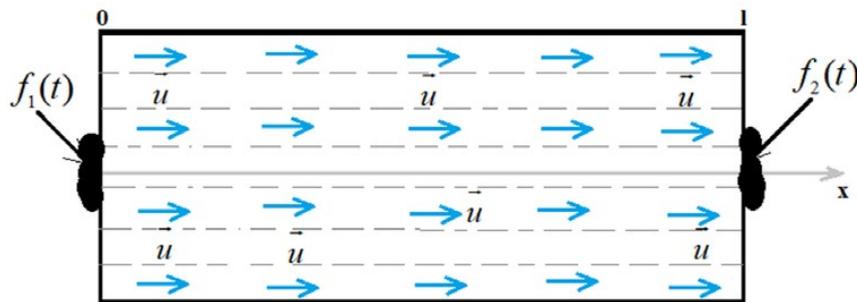


Fig. 1. Graphical representation of the problem

Generally, the analytical solution of the ADE defined in Eqs. (5)-(8) is obtained using the GITT. To apply the GITT, it is necessary to homogenize the boundary conditions as the solutions of non-homogeneous problems based on eigen-function expansions may converge slowly or even exhibit anomalous behaviour, especially in the vicinity of the boundaries (Ozisk, 1980; Almeida & Cotta, 1995; Cotta & Mikhailov, 1997; Liu et al., 2000). The following new independent variable is defined:

$$C(x,t) = f_1(t) + [f_2(t) - f_1(t)] \frac{x}{l} + Q(x,t) \tag{9}$$

where $Q(x,t)$ is the solution of the following problem:

$$\begin{aligned} R \frac{\partial Q}{\partial t} = & D_{x_0} (a_1 + a_2 x)^n \frac{\partial^2 Q}{\partial x^2} - [u_0(a_1 + a_2 x) - na_2 D_{x_0} (a_1 + a_2 x)^{n-1}] \frac{\partial Q}{\partial x} - (u_0 a_2 + \mu_0) Q + q_0 p(x) \\ & - \frac{[f_2(t) - f_1(t)]}{l} [u_0(a_1 + a_2 x) - na_2 D_{x_0} (a_1 + a_2 x)^{n-1}] - (\mu + u_0 a_2) \left\{ f_1(t) + [f_2(t) - f_1(t)] \frac{x}{l} \right\} \\ & - R \left[\frac{df_1(t)}{dt} + \left(\frac{df_2(t)}{dt} - \frac{df_1(t)}{dt} \right) \frac{x}{l} \right] \end{aligned} \tag{10}$$

$Q(x,t)$ have the same forms of boundary conditions as Eqs. (7) and (8) but with the right side now set equal to zero for both equations.

The initial condition of $Q(x,t)$ becomes

$$Q(x,t) = h(x) + K_s^i - f_1(0) - [f_2(0) - f_1(0)] \frac{x}{l}; \quad x \geq 0, \quad t = 0 \tag{11}$$

When solving problem using the GITT, a pair of transforms, namely an integral transform and an inverse transform, has to be established (Almeida & Cotta, 1995; Cotta, 1993; Liu et al., 1998; Suk, 2013). The auxiliary problem must be chosen such that constructing the pair of transforms be simplified and that the solution converges for much lesser number of terms. Bharati et al. (2017) and Pérez Guerrero et al. (2009) proposed to use an eigenvalue problem of SLP with self-adjoin second order operator. The regular SLP chosen for this study is the same as that proposed by (Bharati et al., 2017; Bharati et al., 2018; Bharati et al., 2019) which has the particularity of making concentration converge for the first five number of summation. The selected ODE of the SLP is written as:

$$\frac{d}{dx} \left(e^x \frac{d\varphi}{dx} \right) + (1 + \beta) e^x \varphi = 0 \tag{12}$$

with the following associated two homogeneous first type boundary conditions:

$$\varphi(x = 0) = 0 \tag{13}$$

$$\varphi(x = l) = 0 \tag{14}$$

The trivial solution of this problem is $\varphi = 0$. The nontrivial solutions are called the eigenfunctions belonging to each eigen values β_m , and may be expressed as (Bharati et al., 2017):

$$\varphi_m(x) = \exp(-x/2l) \sin \left[\left(\sqrt{3 + 4\beta_m} / 2 \right) x \right] \quad (15)$$

where the eigen values β_m are given by $\beta_m = \frac{m^2 \pi^2}{l^2} - \frac{3}{4}$ and $m = 1, 2, 3, 4, \dots$

The orthogonality property for the set of linearly independent eigen functions $\varphi_m(x)$ reference to the weight function $\rho(x) = e^x$, associated with Eq. (12) is given by:

$$\int_0^l e^{x/l} \varphi_m(x) \varphi_k(x) dx = \delta_{mk} N_m \quad (16)$$

where N_m is the norm and δ_{mk} is the Kronecker delta. The norm and the normalized eigen functions $\psi_m(x)$ are respectively given by:

$$N_m = \int_0^l e^{x/l} \varphi_m(x) \varphi_m(x) dx = \frac{l}{2} \quad (17)$$

$$\psi_m(x) = \frac{\varphi_m(x)}{\sqrt{N_m}} \quad (18)$$

Now, the unknown function $Q(x, t)$ is represented as a series expansion in terms of the normalized eigen functions $\psi_m(x)$ as:

$$Q(x, t) = \sum_{m=1}^{M \rightarrow \infty} \psi_m(x) T_m(t) \quad (\text{Inverse}) \quad (19)$$

where $T_m(t)$ is the transformed "potential". Eq. (19) is the inverse transform rule. The corresponding transform rule is obtained by following the procedure of Cotta (1993) and Ozisik (1993) i.e. applying the operator $\int_0^l \rho(x) \psi_m(x) (\bullet) dx$ to both sides of Eq. (10) and using Eq. (19) (the orthogonality property) and Eq. (20) to obtain

$$T_m(t) = \int_0^l \rho(x) \psi_m(x) Q(x, t) dx \quad (\text{Transform}) \quad (20)$$

Substituting this solution in the ADE in Eq. (10), multiplying by $e^x \psi_m(x)$, integrating over the given domain, and using the orthogonality properties in Eq. (17), the result is a system of first order ordinary differential equations, a system of IVP, in matrix form, as:

$$\mathbf{A} \frac{d\mathbf{T}_m(t)}{dt} + \mathbf{B}\mathbf{T}_m(t) = \mathbf{G}_m(t) \quad (21)$$

with the transform initial condition

$$\mathbf{T}_m(0) = \sqrt{2} \int_0^l \left\{ h(x) + K_s^i - f_1(0) - [f_2(0) - f_1(0)] \frac{x}{l} \right\} e^{x/2l} \sin\left(m\pi \frac{x}{l}\right) dx \tag{22}$$

where elements of the $M \times M$ matrices \mathbf{A} and \mathbf{B} and the M -length vectors \mathbf{G} are given by:

$$A_{m,k} = \frac{2}{l} R \int_0^l \sin\left(m\pi \frac{x}{l}\right) \sin\left(k\pi \frac{x}{l}\right) dx \tag{23}$$

$$\begin{aligned} B_{m,k} = & \frac{2}{l} \int_0^l D_{x_0} (a_1 + a_2 x)^n \left[\left(\frac{m^2 \pi^2}{l^2} - \frac{1}{4l^2} \right) \sin\left(m\pi \frac{x}{l}\right) + \frac{m\pi}{l^2} \cos\left(m\pi \frac{x}{l}\right) \right] \sin\left(k\pi \frac{x}{l}\right) dx \\ & + \frac{2}{l} \int_0^l \left[u_0(a_1 + a_2 x) - na_2 D_{x_0} (a_1 + a_2 x)^{n-1} \right] \left[\frac{m\pi}{l} \cos\left(m\pi \frac{x}{l}\right) - \frac{1}{2l} \sin\left(m\pi \frac{x}{l}\right) \right] \sin\left(k\pi \frac{x}{l}\right) dx \\ & + \frac{2}{l} \int_0^l (u_0 a_2 + \mu) \sin\left(m\pi \frac{x}{l}\right) \sin\left(k\pi \frac{x}{l}\right) dx \end{aligned} \tag{24}$$

$$\begin{aligned} G_m(t) = & \sqrt{\frac{2}{l}} \int_0^l \left\{ q(t)p(x) - \frac{(f_2(t) - f_1(t))}{l} \left[u_0(a_1 + a_2 x) - na_2 D_{x_0} (a_1 + a_2 x)^{n-1} \right] - (u_0 a_2 + \mu) \left[f_1(t) + (f_2(t) - f_1(t)) \frac{x}{l} \right] \right. \\ & \left. - R \left[\frac{df_1(t)}{dt} + \left(\frac{df_2(t)}{dt} - \frac{df_1(t)}{dt} \right) \frac{x}{l} \right] \sin\left(m\pi \frac{x}{l}\right) e^{\frac{x}{2l}} \right\} dx \end{aligned} \tag{25}$$

The Fourier coefficients in Eq. (21) are given by:

$$\mathbf{T}_m(t) = \exp(-\mathbf{A}^{-1}\mathbf{B}t) \mathbf{T}_m(0) + \exp(-\mathbf{A}^{-1}\mathbf{B}t) \times \int_0^t \exp(\mathbf{A}^{-1}\mathbf{B}\tau) \mathbf{A}^{-1} \mathbf{G}(\tau) d\tau \tag{26}$$

The expression of the Fourier coefficients in Eq. (26) is a particular form of that in Eq. (17) of Liu et al. (2000).

Analytical solutions are obtained in Euclidean and fractal framework i.e for $n = 1, 1.5$ and 2 , representing an index of the spatial dependence of the dispersion coefficient in the dispersion theory $D \propto u^n$ as proposed by Freeze & Cherry (1979). As Bharati et al. (2017), Bharati et al. (2018) and Bharati et al. (2019), we found that our analytical solution converges to the designer pattern with the first five terms ($N = 5$) of the Fourier series. Thus, the solution in Fourier series with first five terms may be written as:

$$\begin{aligned} C(x,t) = & f_1(t) + [f_2(t) - f_1(t)] \frac{x}{l} + \sqrt{2} \exp(-x/2) \left[T_1(t) \sin\left(\pi \frac{x}{l}\right) + T_2(t) \sin\left(2\pi \frac{x}{l}\right) + T_3(t) \sin\left(3\pi \frac{x}{l}\right) \right. \\ & \left. + T_4(t) \sin\left(4\pi \frac{x}{l}\right) + T_5(t) \sin\left(5\pi \frac{x}{l}\right) \right] \end{aligned} \tag{27}$$

RESULTS AND DISCUSSION

Verification of the solution

The solute concentration pattern due to two continuous point sources is obtained from the proposed analytical solutions in groundwater finite flow domain and finite temporal domain as mentioned in the previous section. For short special domain, Chaudhary et al. (2020) proposed to use $\exp(-\sec(\lambda x))$ as distribution of background concentration in the liquid phase because, its decreasing rate is much slower than exponential and could well represent slow movement of groundwater.

The accuracy of the analytical solutions is evaluated using the following solute transport problem

$$R \frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left[D_{x_0} (a_1 + a_2 x)^n \frac{\partial C}{\partial x} - u_0 (a_1 + a_2 x) \right] - \mu C + \gamma_0 p(x) q(t) \quad (28)$$

$$C(x, t) = C_i \exp[-\sec(\lambda x)] + K_s^i; \quad x, \geq 0, \quad t = 0 \quad (29)$$

$$C(x, t) = C_1 \exp(-\lambda_1 t), \quad x = 0, \quad t > 0 \quad (30)$$

$$C(x, t) = C_2 \exp(-\lambda_1 t), \quad x = l, \quad t > 0 \quad (31)$$

where $C_i [ML^{-3}]$ is the uniform input liquid phase concentration, $C [ML^{-3}]$ and $C_2 [ML^{-3}]$ are the uniform sources concentration at the origin and end of the domain respectively, and $\lambda_1 [T^{-1}]$ is the flow resistance coefficient.

We considered a finite spatial and temporal domain defined as $0 \leq x(m) \leq 1$ and $0 \leq t(year) \leq 7$. The solute concentration strength is evaluated from the analytical solutions given in Eq. (27) and the five time-dependent Fourier coefficients are given by Eq. (20). If the solute distribution coefficient K_d is negligible (i.e. $K_d \approx 0$), there is no interaction between the solute and soil, and the retardation factor R becomes equal to water content θ , which is less than 1. So, the retardation factor R becomes less than 1 for the case mentioned above. This indicates that only a fraction of liquid phase concentration participates in the transport mechanism. The input parameters values used are given by Singh & Kumari (2014) $C_1 = 1.0 \text{ mg/L}$, $C_i = 0.01 \text{ mg/L}$, $K_s^i = 0.01 \text{ mg/L}$, $u_0 = 0.01 \text{ m/year}$, $D_0 = 0.01 \text{ m}^2/\text{year}$, $a_1 = 1$, $D^* = 0.002 \text{ m}^2/\text{year}$, $f = 0.8$, $K_d = 0.01$, $s = 0.01 \text{ year}^{-1}$, $\mu_l = 0.0027 \text{ year}^{-1}$, $\mu_s = 0.13 \text{ year}^{-1}$, $a_2 = 0.8 \text{ m}^{-1}$, $\gamma_0 = 0.02 \text{ mg/L year}$. The uniform input concentration in the end of the domain is set to be $C_2 = 0.5 \text{ mg/L}$. The space variation function of the source term is also considered as $a = 3 \text{ m}^{-1}$. Three geological formations are considered here with average porosity η and bulk density ρ_b as follows (Manger, 1963; Freeze & Cherry, 1979): $\eta = 0.3$ (sandstone), 0.1 (shale), 0.5 (gravel); $\rho_b = 2.49$ (sandstone), 2.39 (shale), 2.68 (gravel).

Figures (2), (3) and (4) plot the curves showing the solute concentration distribution in the reservoir for the three types of geological formations at $t = 1, 2, 4$ and 6 years for $n = 1$, $n = 1.5$ and $n = 2$ respectively. The solid lines represent the curves of analytical solutions and the circles symbol, those of numerical solutions. It is found that for each value of n and both geological formations, at $t = 1 \text{ year}$ in this example, the solute profile is qualitatively similar to that predicted with standard advection-dispersion models. After a certain time, the solute concentration increases with traveling distance from approximative value of 1.0 mg/L at $x = 0 \text{ m}$ to a maximum value of concentration, then decreases back to 0.5 mg/L at the end of

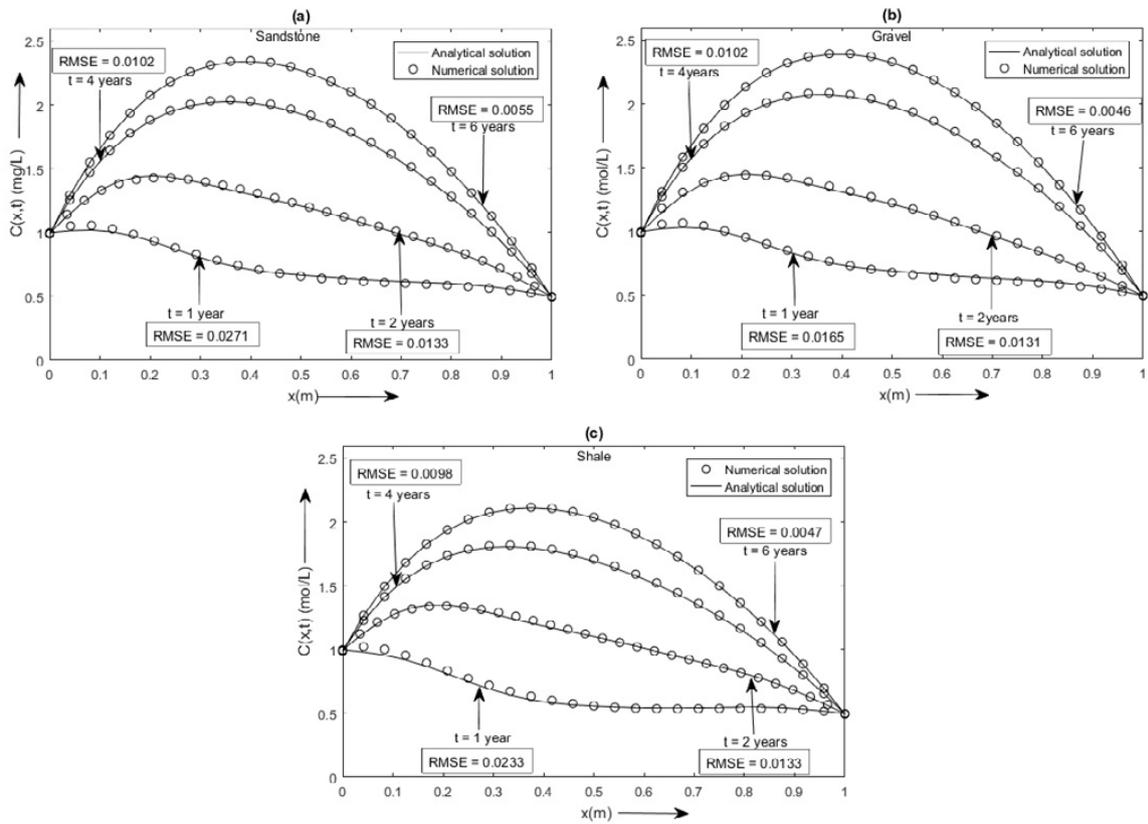


Fig. 2. Comparison of analytical and numerical solutions for contaminant transport with two input sources at four different times for $n = 1$ obtained for different geological formations.

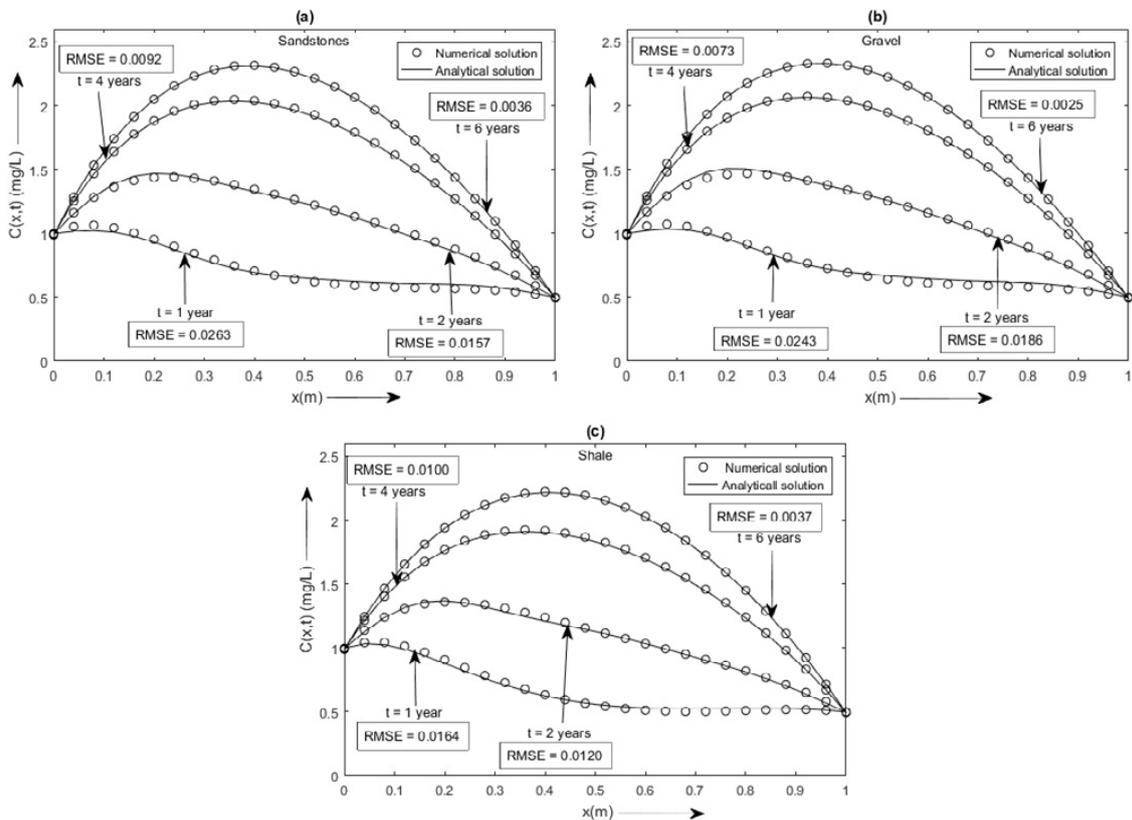


Fig. 3. Comparison of analytical and numerical solutions for contaminant transport with two input sources at four different times for $n = 1.5$ obtained for different geological formations.

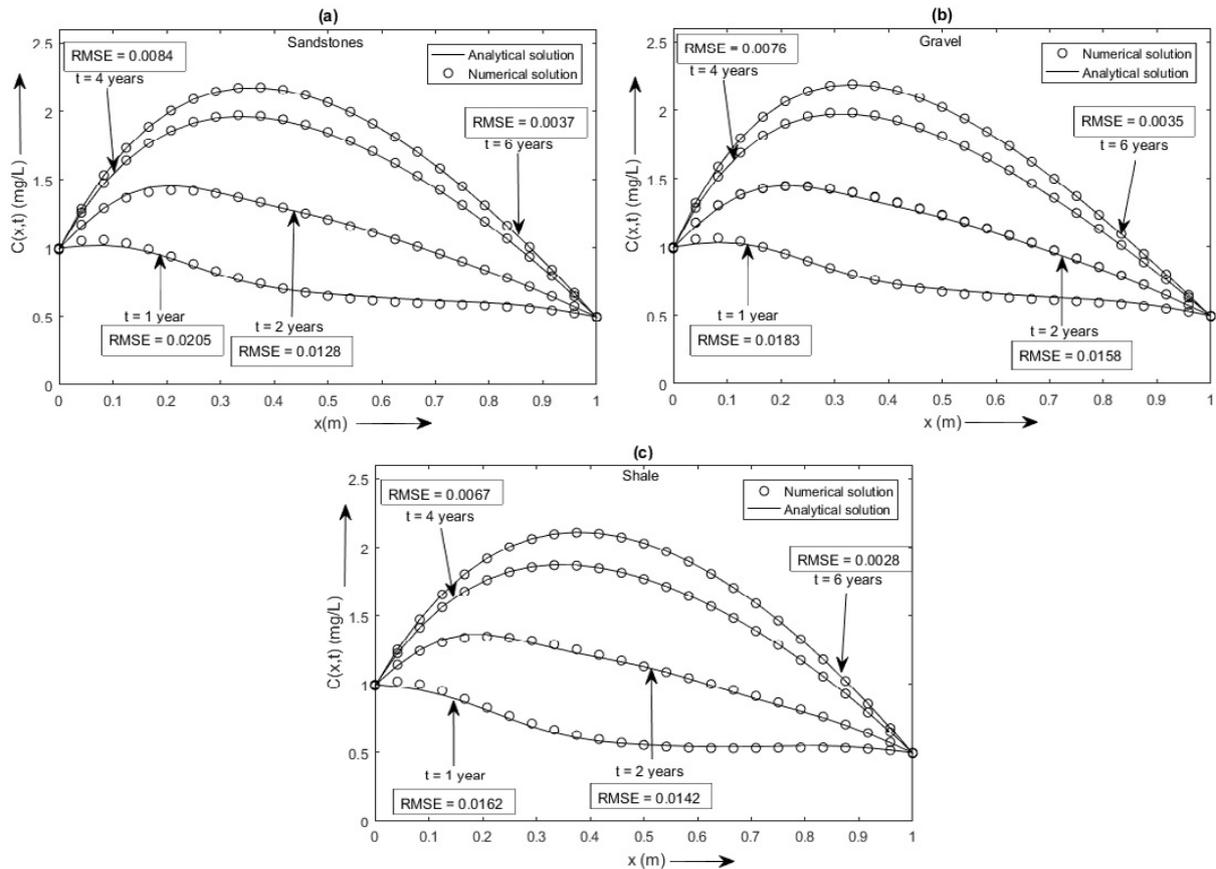


Fig. 4. Comparison of analytical and numerical solutions for contaminant transport with two input sources at four different times for $n = 2$ obtained for different geological formations.

the domain. This build-up of concentration is due to the concentration effect of the additional source term in the reservoir that decreases with distance. The solute build-up depends on the geological formation and on the value of n , for a fixed time. For example, a comparison of Figs. 2(a)-4(c) show that the peak of concentration decreases with the increasing value of n in both geological sites. The distance concerned by the solute build-up is very important and increases with increasing time, also, the maximum of concentration value gradually increases with increasing time. The solute build-up in a large distance is due to the presence of two contaminant sources, each acting separately at the boundary of the reservoir.

The analytical solutions in each case are also compared with the respective numerical solutions obtained using MATLAB *pdepe* solver. The results show that in each geological formation, both solutions are in good agreement in both forms of dispersity, as illustrated by the curves and the value of Root Mean Square Error (RMSE). Hence, the higher values of RMSE are observed for $n = 1.5$. It can be concluded that analytical solutions converge well with five terms of summation.

Pollutant concentrations in both geological formations are compared for each value of n in Figure 5. The curves are plotted at $t = 2$ years and $t = 4$ years with the same input as in Figs. (2)-(4). Figure 5 depicts that for $n = 1$ and 1.5 , the concentration level is higher in gravel with bulk density ($\rho_b = 2.68$) compared to sandstone ($\rho_b = 2.49$) and shale ($\rho_b = 2.39$) at each of the position and time. While for $n = 2$ at the fixed time $t = 2$ years, the concentration values of gravel formation are higher at each of the position for this domain i.e., $0 \leq x \leq 0.2$ m but from 0.2 m to the end of the domain, the concentration values are higher for the sandstone formation. At this time, the shale geological formation has the lower values of pollutant concentration in all the domain. At fixed time $t = 4$ years, the pollutant concentration values are higher in

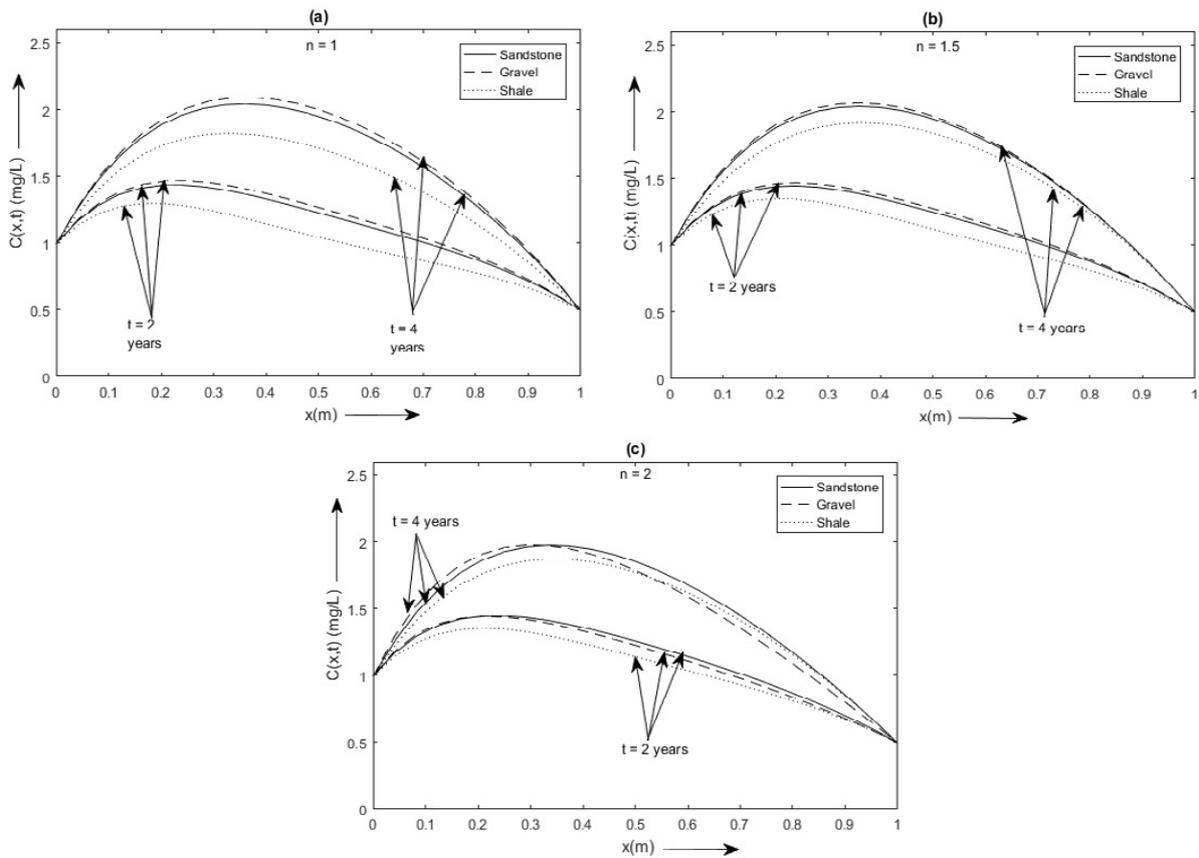


Fig. 5. Comparison of solute concentration at different times for different geological formations

gravel in comparison to sandstone for $0 \leq x \leq 0.35 \text{ m}$ and than shale formation for this domain $0 \leq x \leq 0.55 \text{ m}$. At this time, the concentration values in sandstone are higher than those of shale formation at each of the position of the domain. We also note that when n increases, the difference on pollutant concentration levels in both formation decreases.

Illustration of the model

In the last decades, human activities and industrial development of industrialization these last decades have increased considerably the number of pollution sources in the environment, particularly in the groundwater. Some contaminants wastes are evacuated by septic tanks, other substances as petroleum are transported via pipes. These sources can be cause multiple contaminations of a medium in different positions. Although literature contains many studies referring to multiples contaminants input sources, few of them considered the effect of a variety of boundary inputs (e.g. Mazarheti, 2013). But no one has considered the input sources localized at the origin and end of the domain. The analytical solutions obtained in this study are illustrated for contaminant transport with source-production decay which have many real world hydrological applications and are large importance in soil contamination. This is the case of sequential decay of multi-species contaminants as nitrogen, chlorinated solvent, and radionuclide.

Figure 6 illustrates the pollutant distribution at $t = 1, 3, 5, 10$ and 50 years in heterogeneous medium for $n = 1.5$. Contaminant concentration pattern is investigated for sandstone geological formation ($\eta = 0.3, \rho_b = 2.49$). Five pollutant input distributions with production-decay are used to illustrate the effect of input source to the solute concentration pattern. The first one considers

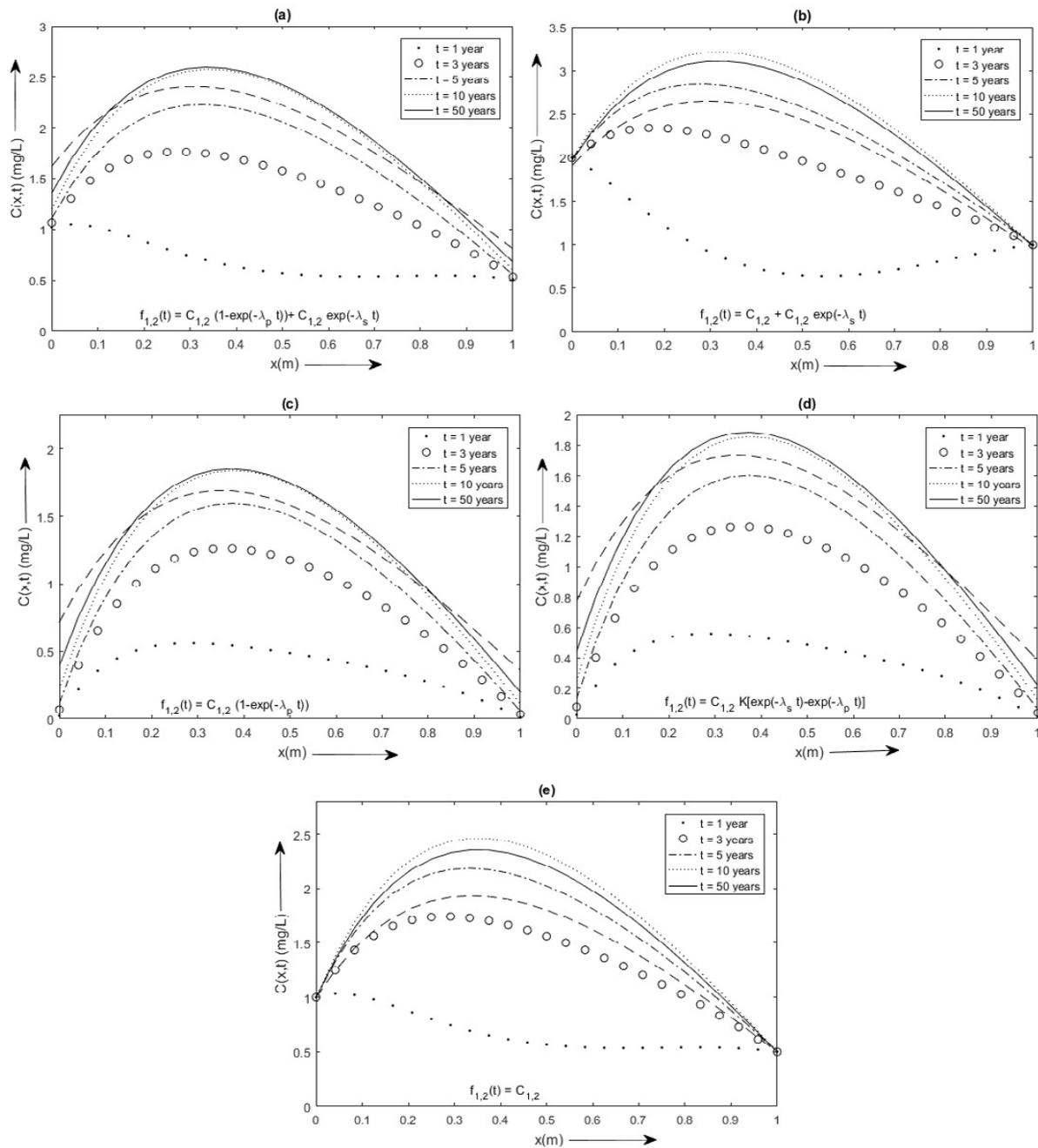


Fig. 6. Solute distribution pattern in sandstone formation for different input distributions for $n = 1.5$

the linear combination of exponential inlet distribution ($f_{1,2}(t) = C_{1,2}(1 - e^{-\lambda_p t}) + C_{1,2} e^{-\lambda_s t}$), where $\lambda_p [T^{-1}]$ and $\lambda_s [T^{-1}]$ represent the production and decay constant respectively. For this input distribution, it is possible to obtain well known input distributions for particular values of the production/decay constants.

- Case 1: For $\lambda_p = 0$ and $\lambda_s \rightarrow \infty$ we have $f_{1,2}(t) = C_{1,2}$
- Case 2 : For $\lambda_p \rightarrow \infty$, we obtain $f_{1,2}(t) = C_{1,2} + C_{1,2} e^{-\lambda_s t}$
- Case 3 : For $\lambda_p \rightarrow 0$, we obtain $f_{1,2}(t) = C_{1,2}(1 - e^{-\lambda_p t})$

The last input distribution investigated consider a consecutive reaction at the source given by $f_{1,2}(t) = C_{1,2}K(e^{-\lambda_s t} - e^{-\lambda_p t})$, where K represents the global kinetic rate. This solution can be particularly useful for waste radioactive decay at the source or PCE (tetrachloroethylene) to TCE (trichloroethylene) degradation in soils (Moranda, 2018).

The input parameters values used, and the other distribution function are the same as in Figure (3). Additionally, the following values are taken for decay and production constants $\lambda_p = 0.025 \text{ year}^{-1}$, $\lambda_s = 0.002 \text{ year}^{-1}$ and the kinetic rate $K = 1.25$.

It is clearly observed that, at $t = 1 \text{ year}$ for this example, the pollutant concentration strength exhibits different patterns for the various inputs boundary distribution. But after certain a period, the pollutant concentration starts with a value imposed by the input at the first end of the reservoir i.e., $x = 0$ and concentration increased with one additional source function in the aquifer. The maximum value of pollutant concentration moved from the additional source function; after the contaminant concentration decreased with distance to the concentration value at the end of the domain given by the input distribution at this position. However, the higher pollutant concentration values are observed for the input distribution $f(t) = C + C e^{-\lambda_s t}$ and the lower values for $f(t) = C(1 - e^{-\lambda_p t})$. The figure clearly demonstrate that the pollutant concentration strength is significantly affected by the input boundary distributions.

Figures 7.a and 7.b show the concentration breakthrough curves for $n = 1.5$ at $x = 0.35 \text{ m}$ and $x = 0.8 \text{ m}$ respectively for the different input distributions. The input parameters remain the same as in Figure (3). It is observed that at these positions, the concentration in both distributions increases sharply from zero value at $t = 0$ up to a maximum value reached between 5 and 10 years, and it starts decreasing gradually with time. The increasing and the decreasing rates depend on the pollutant input distribution. For example, we can observe that the increasing and decreasing rates are higher for $f(t) = C + C e^{-\lambda_s t}$ in comparison with other distributions. For $f(t) = C(1 - e^{-\lambda_p t})$, the pollutant concentration increases slowly compared to the other distributions, and the decreasing rate is very low. However, the higher concentration values are observed at $x = 0.35 \text{ m}$. These curves again clearly illustrate that the concentration strength is significantly affected by the form of the input distribution.

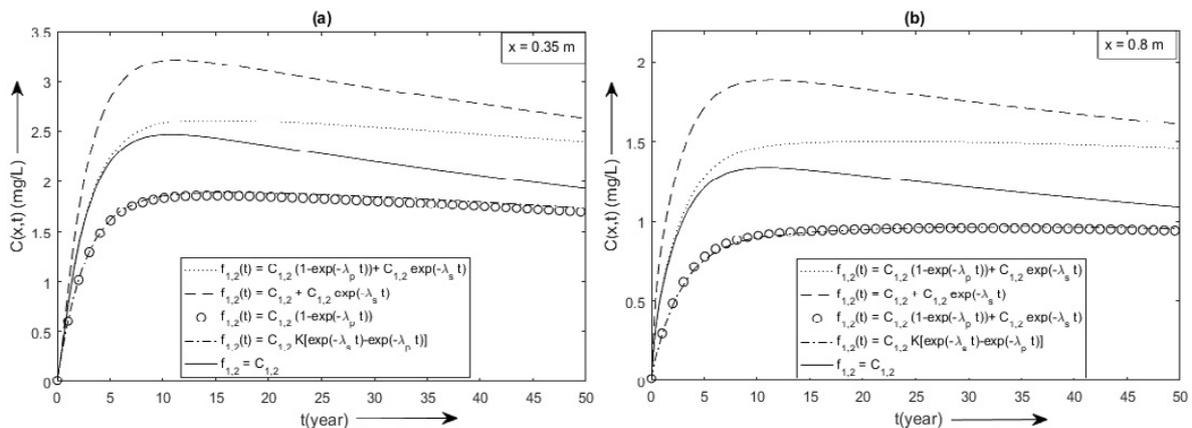


Fig. 7. Breakthrough curves obtained in sandstone formation for $n = 1.5$ for different input distributions.

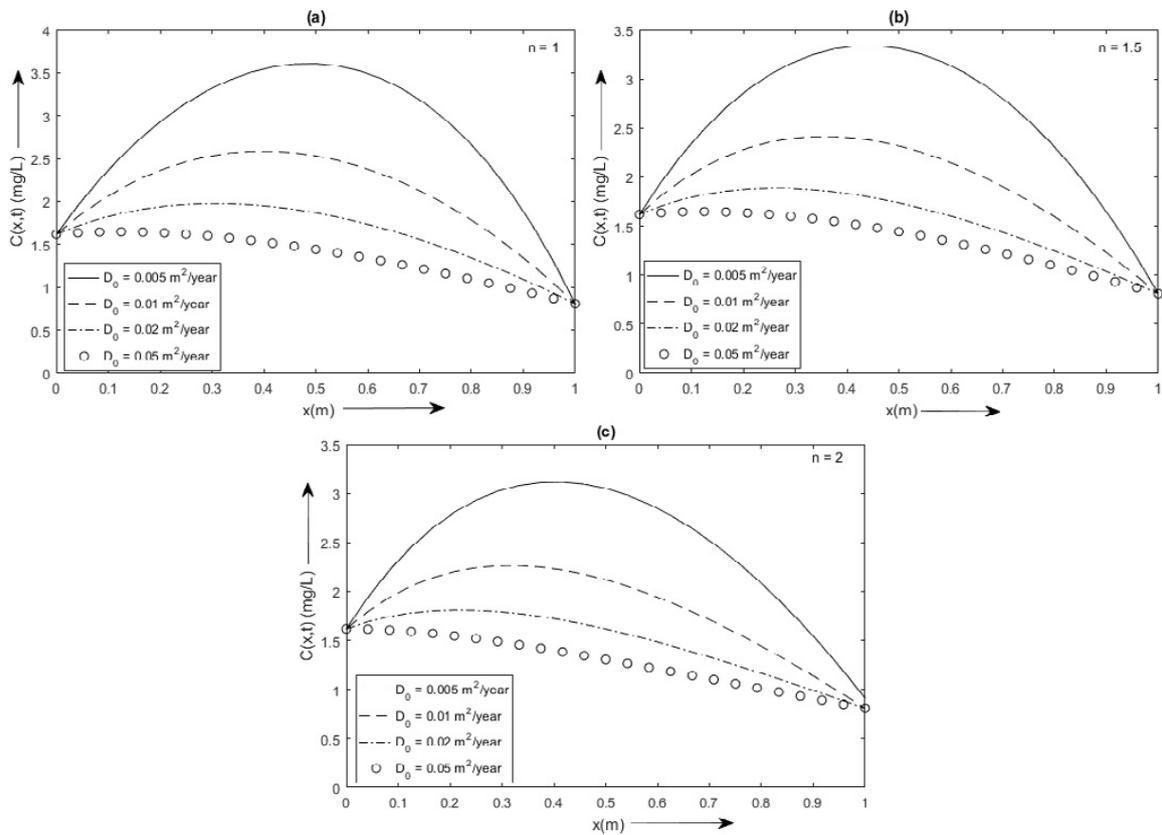


Fig. 8. Effect of dispersion coefficient on solute concentration distribution in presence of two sources localized at $x = 0$ and $x = 1$ km.

The pollutant concentration profile is depicted for four different dispersion profiles ($D_0 = 0.05$, 0.02 , 0.01 and $0.005 \text{ m}^2 / \text{year}$) with fixed velocity and particular time $t = 50 \text{ years}$ as shown in the Fig.8. Figs. 8(a), 8(b) and 8(c) demonstrate this strength for $n = 1$, 1.5 and 2 respectively.

The linear combination of exponential inlet distribution is considered with input parameters remaining the same as in Figs. 2, 3 and 4 respectively. The curves show that, for both values of dispersion coefficient, the pollutant concentration was approximated as $1.65 \text{ mg} / \text{L}$ at the inlet location of the reservoir and concentration increased to a maximum value depending on the value of n because of additional source function. After, the pollutant concentration decreases when distance increases until reaching a fixed concentration value at the end of the reservoir. The peak concentration in each case is large because of the presence of the two contamination sources. The increasing rate is higher for lower values of dispersion coefficient in comparison with higher values of dispersion coefficient while, the decreasing rate is lower for higher values of dispersion coefficient in comparison to lower values of dispersion. For a fixed value of n , the concentration values at each of the intermediate position increases with the decreasing value of the dispersion coefficient. Increasing the dispersion coefficient will decrease the maximum value of solute concentration. Overall, the peak pollutant concentration for various profiles in the front is large, moved from one additional source function and decreases towards the exit boundary. Also, the concentration values at intermediate positions depend on the dispersion coefficient as well as on the value of n . It is clear that the dispersion process plays an important role for the determination of the concentration distribution in the presence of two input sources.

Figure 9 elucidates the effect of time dependence of production term on the concentration pattern in the medium with square root dispersivity ($n = 1.5$) in sandstone geological formation

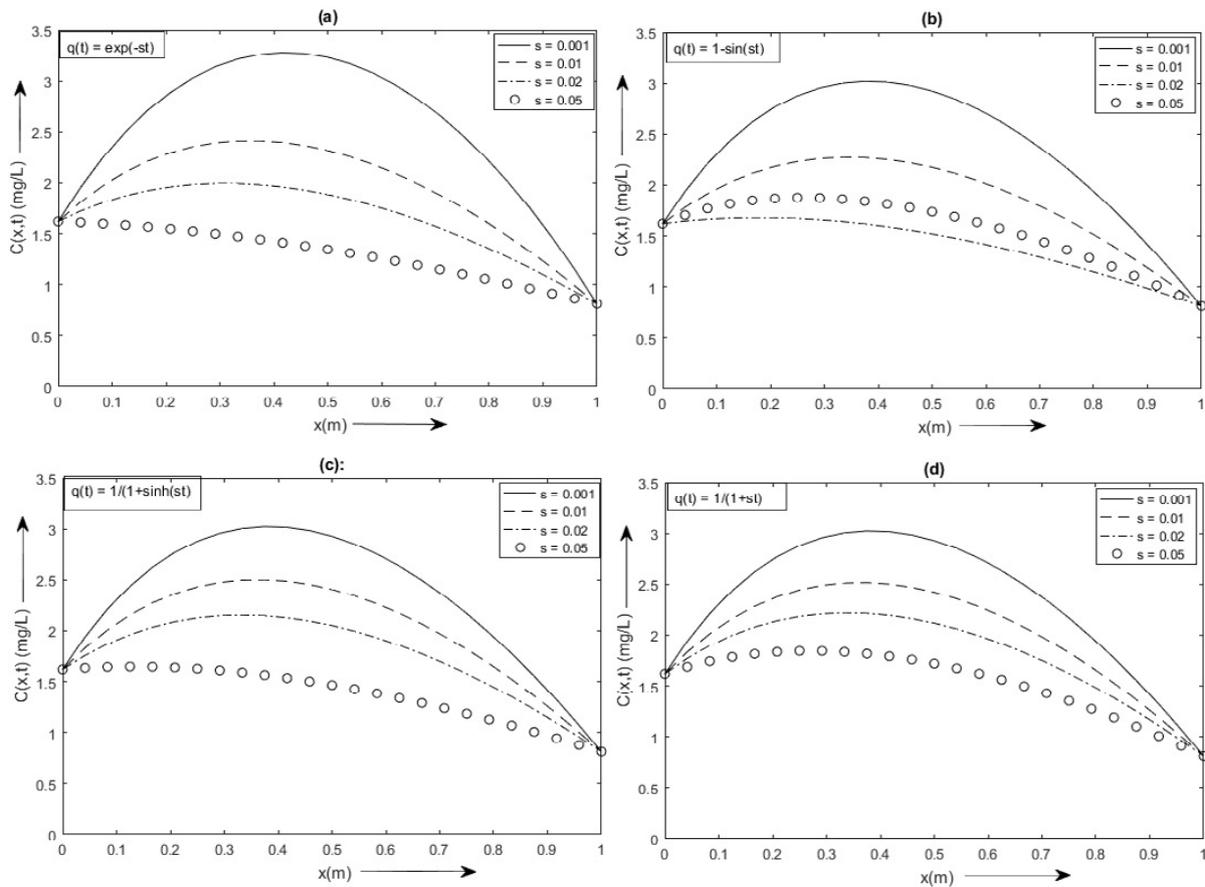


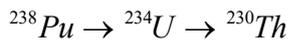
Fig. 9. Effect of source term and unsteady parameter on contaminant concentration

for a fixed time $t = 5$ years. Four expressions of time decreasing functions for the production term are used to illustrate this effect; the exponential decreasing function ($q(t) = \exp(-st)$), a sinusoidal function ($q(t) = 1 - \sin(st)$), an hyperbolic function ($q(t) = \frac{1}{1 + \sinh(st)}$) and $q(t) = \frac{1}{1 + st}$. The solute concentration profiles are obtained with the same parameters values as in Fig. (3.a). For all the profiles, the concentration starts with a constant value at the origin of the domain, increases with distance to a maximum value, then decreases to a minimum value at the end of the domain. The pollutant concentration values at the boundary of the domain depend only on the expression of the time dependent function. The figure depicts that the concentration level at different positions is attenuated with the increasing value of the parameter s in both the expressions of time dependent source term except for the sinusoidal function. For this function, the concentration level decreases with the increasing value of s until a certain value and then begins to increase with the increasing value of s . This result can be attributed to the periodical behavior of the sinusoidal function, causing the concentration level to depend to the frequency s of the sinusoidal function. The curves depicts also that the variation of the parameter s produces an important variation on pollutant concentration level in the case of sinusoidal and exponential decaying functions in comparison with the other.

Example of application

Many contaminant sources are present in the environment and some of them are subject of studies due to their importance or the damage they can cause to living organisms. These contaminant sources are of various nature and origin, among which the radionuclides. Several

radionuclides are members of a radionuclide decay chain. A lot of researchers have developed various models involving sets of advective-dispersive transport equations coupled by first-order decay (van Genuchten, 1985; Suk, 2013; Chen et al., 2019; Yu et al., 2019) but analytical solutions in closed form that consider both production and decay at the source are scarcely available (Paladino et al., 2018; Moranda et al., 2018; Sanskrityayn & Kumar, 2017). To illustrate the model, the following example of application has been considered: a transport of a second member of radionuclide decay chain during its movement into groundwater with two sources



In the chosen example, the radionuclide source undergoes at the same time a decay and a production so that the boundary functions in Eqs. (7) and (8) are written as

$$f_1(t) = C_1 K (e^{-\lambda_s t} - e^{-\lambda_p t}) \quad (32)$$

$$f_2(t) = C_2 K (e^{-\lambda_s t} - e^{-\lambda_p t}) \quad (33)$$

where λ_s and λ_p are respectively the decay and production constant [T^{-1}], K is the kinetic rate.

When studying the distribution of radionuclides in a medium, we are most often interested in their environmental impact, especially in the effective dose that human beings can absorb. Based on the studied model, one can determine the radionuclide absorbing dose from ingestion of drinking water.

The committed effective dose per person from a given radionuclide through groundwater can be calculated by:

$$\text{Committed effective dose} = IR \times C \times DF \quad (34)$$

where IR is the rate of intake (m^3 / day), C is the radioactivity concentration in groundwater of the nuclide (Bq / m^3), and DF is the ingestion dose coefficient of the nuclide for the adult age group (Sv / Bq).

Input parameters, used here, are kept from (Carntrell et al., 2003; ICRP, 2012; Chen et al., 2019; Chaudhary et al., 2020) except for input concentration value and, they are summarized in Table 1.

Figure 10.a depicts the radionuclide concentration as a function of time at different distances from the origin ($x = 50$, $x = 100$ and $x = 200$ m) for different geological formations. The concentration of ^{234}U at each of the position increases in the early time period and starts decaying after a certain year depending on the geological formation and the position. However, the concentration of ^{234}U maintains the increasing trends up to 10,000 years. The gravel formation has the higher radioactive concentration than the two others geological formations. The highest value of concentration is observed for $x = 200$ m at $t = 1,000$ years with the value of 2.1×10^9 Bq / m^3 . It could be seen that the radioactivity concentration level decreases with position and then increases due to the second radioactive source localized at the end of the domain.

Figure 10.b depicts the time history of radionuclide dose at different distances from the origin ($x = 50$, $x = 100$ and $x = 200$ m) for different geological formations. A comparison of Figs. 10.a with 10.b shows that the amplitude of the doses at different positions and for each geological formation follows the same sequence of the magnitude of the corresponding concentration. As the radioactive concentration, the higher dose values are obtained for gravel formation for an

Table 1. Input parameters for uranium

Parameters	Units	Values
Water content, θ	m^3 / m^3	0.4
Distribution coefficient, K_d	m^3 / kg	0.2
Porosity, η	-	Sandstone, 0.3
		Shale, 0.1
		Gravel, 0.5
Bulk density, ρ_b	Kg / m^3	Sandstone, 2490
		Shale, 2680
		Gravel, 2390
Effective dispersion coefficient, D_0	$m^2 / year$	1000
Groundwater velocity, u_0	$m / year$	100
Radioactive decay constant, μ	$year^{-1}$	0.0000028
Production constant, λ_p	$year^{-1}$	0.0089
Source decay constant, λ_s	$year^{-1}$	0.0010028
Initial amount of radionuclide, C_1	Bq / m^2	10^{15}
Initial amount of radionuclide, C_2	Bq / m^2	5×10^{14}
Rate of intake, IR	m^3 / day	0.005
Ingestion dose coefficient, DF	Sv / Bq	4.9×10^{-8}

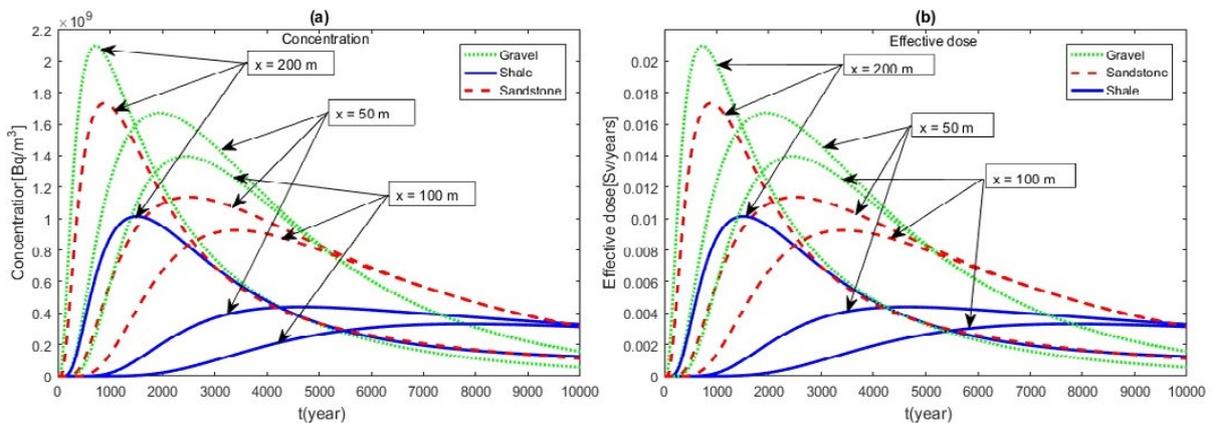


Fig. 10. (a) Time history radionuclide at different distances from the origin ; (b) Effective dose of radionuclide acquired through drinking groundwater pathway at different distances from the origin

adult located at $x = 200\ m$ from the origin. At that location, the highest value of the amplitude of the dose is about $0.021\ Sv / year$.

The smallest value of the dose is obtained for the shale formation at $100\ m$ of the origin. However, for both geological formations and at both positions, the annual doses are all above the WHO guideline of $0.1\ mSv/year$ for the drinking water pathway. The generalized analytical solutions can quickly and accurately predict the one-dimensional solute (as radionuclide) migration and assess the radiological impact posed by radionuclides in the environment as a result of leakages from a nuclear waste repository or accidental discharge from a nuclear facility.

COCLUSION

An analytical solution of 1-D ADE solute transport with distance dependent coefficients in a sorbing finite groundwater reservoir with an additional source-sink term was derived. The transport problem considered an hypothetical two continuous pollutant sources localized at the origin and at end of the domain. The velocity was considered as a linear function of space function while the dispersion coefficient was considered as a n^{th} power of the velocity. Analytical solution is derived via the GITT using an advection-dispersion SLP with a new self-adjoint operator. The effect of some parameters on the equation was investigated with the help of graphs. The results show that for such situations, the concentration increases in the direction of flow, reaching a peak of concentration, then decreases with distance because of the additional source function which decreases with position. The peak concentration is large due to the presence of two pollutant input sources at the boundary of the reservoir. The concentration levels in the reservoir depend on the geological formation and the degree of heterogeneity (value of n), with the highest value obtained for $n = 1$. Overall, by varying the input boundary conditions, the values of n , the dispersion coefficient, the value of uniform source term and the geological formation a set of concentration profile can be generated. The accuracy of the calculated analytical contaminant strength is analysed with their corresponding numerical results obtained by MATLAB *pdepe Solver*, which were found in acceptable compliance with each other for both values of n . The obtained analytical solution could be useful for estimating the transport of contaminant in heterogeneous and homogeneous, sorbing groundwater reservoir with two sources of contaminations. Furthermore, it can be recommended as a tool for assessing human risk by drinking water as illustrated by the example of application studied. In the future, we intent to address the problem for the case of unsteady parameters but also to extend the study to the case of several contaminants sources through arbitrary time-dependent emission rate patterns.

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CONFLICT OF INTEREST

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

LIFE SCIENCE REPORTING

No life science threat was practiced in this research.

REFERENCES

Almeida, A. R. and Cotta, R. M. (1995). Integral transform methodology for convection-diffusion problems

- in petroleum reservoir engineering. *Int. J. Heat Mass Transfer*, 38(18); 3359-3367.
- Bharati, V. K., Singh, V. P., Sanskrityayn, A. and Kumar, N. (2017). Analytical solution of advection diffusion equation with spatially dependent dispersivity. *J. Eng. Mech.*, 143(11); 1-11.
- Bharati, V. K., Singh, V. P., Sanskrityayn, A. and Kumar, N. (2018). Analytical solutions for solute transport from varying pulse source along porous media flow with spatial dispersivity in fractal & Euclidean framework. *Eur. J. Mech. B Fluids*, 72; 410-421.
- Bharati, V. K., Singh, V. P., Sanskrityayn, A. and Kumar, N. (2019). Analytical solution for solute transport from a pulse point source along a medium having concave/convex spatial dispersivity within fractal and Euclidean framework. *J. Earth Syst. Sci.*, 128(203).
- Carnotrell, K. J., Serne, R. J. and Last, G. V. (2003). Hanford contaminant distribution coefficient database and users guide. U.S. department of Energy under contract DE-AC06-76RL01830, Pacific Northwest National laboratory Richland Washington 99352. PNNL-13895 Rev. 1.
- Chaudhary, M., Kumar, Thakur, C. and Kumar Singh, M. (2020). Analysis of 1-D pollutant transport in semi-infinite groundwater reservoir. *Environmental Earth Sciences* (2020) 79(24).
- Chen, J. S., Chen, J. T., Liu, C. W., Liang, C. P. and Lin, C. W. (2011). Analytical solutions to two-dimensional advection-dispersion equation in cylindrical coordinates in finite domain subject to first- and third-type inlet boundary conditions. *J. Hydrol.*, 405; 522-531.
- Chen, J. S. and Liu, C. W. (2011). Generalized analytical solution for advection-dispersion equation in finite spatial domain with arbitrary time-dependent inlet boundary condition. *Hydrol. Earth Syst. Sci.*, 15(8); 2471-2479.
- Chen, J. S., Liang, J. P., Chang, C. H. and Ming-Hsien (2019). Wan simulating three-dimensional plume migration of a radionuclide decay chain through groundwater. *Energies*, 2019 12(3740).
- Cotta, R. M. (1993). *Integral Transforms in Computational Heat and Fluid Flow*. CRC Press, Boca Raton, FL.
- Cotta, R. M. and Mikhailov, M. D. (1997). *Heat Conduction: Lumped Analysis, Integral Transforms, Symbolic Computations*. Wiley-Interscience, New York (1997).
- Crank, J. (1975). *The Mathematics of Diffusion*. Oxford Univ. Press London, 2nd ed.
- Freeze, R. A. and Cherry, J. A. (1979). *Groundwater*, Prentice Hall, Englewood Cliffs, NJ.
- ICRP. *Compendium of Dose Coefficients based on ICRP Publication 60*; ICRPP ublication: Ottawa, ON, Canada, 2012.
- Kumar, A. and Yadav, R. R. (2014). One-dimensional solute transport for uniform and varying pulse type input point source through heterogeneous medium. *Environmental technology*.
- Kumar, R., Chatterjee, A., Singh, M. K. and Singh, V. P. (2019). Study of solute dispersion with source/sink impact in semi-infinite porous medium. *Pollution*, 6(1); 87-98.
- Liu, C., Ball, W. P. and Ellis, J. H. (1998). An analytical solution to one-dimensional solute advection-dispersion equation in multi-layer porous media. *Transp. Porous Media*, 30; 25-43.
- Liu, C., Szecsody, J. E., Zachara, J. M. and Ball, W. P. (2000). Use of the generalized integral transform method for solving equations of solute transport in porous media. *Adv. Water Resour.*, 23; 483-492.
- Manger, G. E. (1963). Porosity and bulk density of sedimentary rocks. U.S. Atomic Energy Commission USGPO, Washington, D.C.
- Mazarheti, M. Samani, J. M. V. and Samani, H. M. V. (2013). Analytical solution to one-dimensional advection-diffusion equation with several point sources through arbitrary time-dependent emission rate patterns. *J. Agr. Sci. Tech.*, 15; 1231-1245.
- Moranda, A., Cianci, R. and Paladino, O. (2018). Analytical solutions of one-dimensional contaminant transport in soils with source production-decay. *Soil Syst.*, 2018, 2(40).
- Ozisik, M. N. (1980). *Heat Conduction*. Wiley, New York.
- Ozisik, M. N. (1993). *Heat Conduction*. John Wiley and Sons Inc., New York.
- Pang, L., Hunt, B., 2001. Solutions and verification of a scale-dependent dispersion model. *J. Contam. Hydrol.*, 53 (12); 2139.
- Park, E. and Zhan, H. (2001). Analytical solution of contaminant transport from one-, two-, and three-dimensional sources in a finite-thickness aquifer. *J. Contam. Hydrol.*, 53; 41-61.
- Pérez, Guerrero, J. S., Pimentel, L. C. G., Skaggs, T. H. and van Genuchten, M. Th. (2009). Analytical solution of the advection-diffusion transport equation using a change-of-variable and integral transform technique. *Int. J. Heat Mass Transfer*, 52(1314); 3297-3304.
- Pérez, Guerrero, J. S. and Skaggs, T. H. (2010). Analytical solution for one-dimensional advection-dispersion transport equation with distance-dependent coefficients. *J. Hydrol.*, 390 (2010) 5765.
- Paladino, O., Moranda, A., Massabò, M. and Robbins, G. A. (2018). Analytical Solutions of Three-Dimensional Contaminant Transport Models with Exponential Source Decay. *Groundwater*, 2018 56; 6-108..

- Sanskritayn, A. and Kumar, N. (2016). Analytical solution of advection-diffusion equation in heterogeneous infinite medium using Green's function method. *J. Earth Syst Sci.*, 125(8); 1713-1723.
- Sanskritayn, A., Suk, H. and Kumar, N. (2017). Analytical solutions for solute transport in groundwater and riverine flow using Green's Function Method and pertinent coordinate transformation method. *J. Hydrol.*, 2017, 547; 517-533.
- Sanskritayn, A., Singh, V. P., Bharati, V. K. and Kumar, N. (2018). Analytical solution of two-dimensional advection-dispersion equation with spatio-temporal coefficients for point sources in an infinite medium using Green's function method. *Environ. Fluid Mech.*, 18; 739-757.
- Serrano, S. E. (1992). The form of the dispersion equation under recharge and variable velocity, and its analytical solution. *Water Resour. Res.*, 28(7); 1801-1808.
- Sim, Y. and Chrysikopoulos, C. V. (1996). One-dimensional virus transport in porous media with time-dependent inactivation rate coefficients. *Water Resour. Res.*, 32(8); 2607-2611.
- Singh, M. K. and Kumari, P. (2014). Contaminant concentration prediction along unsteady groundwater flow. In: Basu SK, Kumar N (eds) *Modelling and simulation of diffusive processes*. Springer, Cham, 257-275.
- Singh, M. K. and Das, P. (2015). Scale dependent solute dispersion with linear isotherm in heterogeneous medium. *J. Hydrol.*, 520; 289-299.
- Skaggs, T. H. and Leij, F. J. (2002). Solute Transport: Theoretical Background. In J. H., Dane, C. G. Topp (Eds), *Method of soil analysis, Part 4 Physical methods*, SSSA Books series, 5; 1353-1380, SSSA Madison, WI (Chapter 6.3).
- Skaggs, T. H., Jarvis, N. J., Pontedeiro, E. M., van Genuchten, M. Th. and Cotta, R. M. (2007). Analytical advection-dispersion model for transport and plant uptake of contaminants in the root zone. *Vadose Zone J.*, 6(1).
- Su, N., Sander, G. C., Liu, F. and Anh, Barry, D. A. (2005). Similarity solutions for solute transport in fractal porous media using a time- and scale-dependent dispersivity. *Applied Mathematical Modelling*, 29(9); 852-870.
- Suk, H. (2013). Developing semi-analytical solutions for multispecies transport coupled with a sequential first-order reaction network under variable flow velocities and dispersion coefficients. *Water Resour. Research*, 49; 3044-3048.
- Thakur, C. K., Chaudhary, M., van der Zee S. E. A. T. M. and Singh, M. K. (2019). Two-dimensional solute transport with exponential initial concentration distribution and varying flow velocity. *Pollution*, 5(4); 721-737.
- Van Genuchten, M. Th. (1981). Analytical solutions for chemical transport with simultaneous adsorption, zero-order production and first-order decay. *J. Hydrol.*, 49(3-4); 213-233.
- Van Genuchten, M. Th. and Alves, W. J. (1982). Analytical solutions of the one-dimensional convective-dispersive solute transport equation. *US Dept. Ag. Tech. Bull. N° 1661*; 1-51.
- Van Genuchten, M. Th. (1985). Convective-dispersive transport of solutes involved in sequential first-order decay reactions. *Comput. Geosci.*, 11; 129-147.
- Yadav, R. R. and Kumar, L. K. (2019). Solute transport for pulse type input point source along temporally and spatially dependent flow. *Pollution*, 5(1); 53-70, Winter 2019.
- Yates, S. R. (1992). An analytical solution for one-dimensional transport in porous media with an experimental dispersion function. *Water Resources Research*, 28(8); 2149-2154.
- Yu, C., Zhou, M., Ma, J., Cai, X. and Fang, D. (2019). Application of the homotopy analysis method to multispecies reactive transport equations with general initial conditions. *Hydrogeology Journal*, <https://doi.org/10.1007/s10040-019-01948-7>.
- Zhan, H., Wen, Z., Huang, G. and Sun, D. (2009). Analytical solution of two dimensional solute transports in an aquifer-aquitard system. *J. Contaminant Hydrol.*, 107; 162-174.
- Zoppou, C. and Knight, J. H. (1997). Analytical solutions for advection and advection diffusion equation with spatially variable coefficients. *Journal of Hydraulic Engineering*, 123; 144-148.

